Underwriting Cycle and Ruin Probability

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Abstract

This paper presents a model for analyzing the impact of underwriting cycles on an insurer's surplus. The model allows the insurer to vary its security loading in response to the cycles, with a strategy parameter that indicates the extent to which the insurer follows the loading which prevails in the market. The insurer's claim rate is also allowed to vary to reflect exposure changes that result from the insurer's strategy.

We analyze ruin probabilities using both simulation and a Lundberg-type upper bound which is developed in the paper. We find that the latter is suitable and convenient for comparing ruin probabilities under the different insurer strategies.

Keywords: Periodic risk process; underwriting cycle; upper bound for ruin probability

1 Introduction

The property/casualty insurance market is known to experience underwriting cycles, with a period that tends to be about six years (see Venezian, 1985; Cummins and Outreville, 1987). A soft market occurs when the sum of premium goals for all companies operating in a given market is greater than the amount of insurance desired by all potential insureds in that market. In a soft market, fierce competition forces many insurers' prices below discounted losses and expenses, causing deterioration of loss reserve adequacy and surplus levels, which may lead to insolvency. A hard market occurs when the sum of premium goals for all companies operating in a given market is less than the amount of insurance desired by all potential insureds in that market. In a hard market, insurance prices are high and coverage is difficult to find, causing strengthening of loss reserve adequacy and surplus levels.

The causes and mechanism of underwriting cycles are studied extensively in the insurance literature (see, for example, Venezian, 1985; Cummins and Outreville, 1987; Doherty and Kang, 1988; Harrington and Danzon, 1994; and Doherty and Garven, 1995).

As described in Boor (1998), an insurer may choose various strategies in a cyclic environment:

- **Maintaining Market Share**

  The insurer may follow exactly the profit loading prevailing in the market, no matter how low it is (even a temporarily negative loading). In this way, the insurer will retain a constant number of insureds (exposures) and therefore a constant claim rate.

- **Conserving Capital**

  The insurer does not follow the premium loading prevailing in the market. Instead, it retains a profitable loading θ. In a soft market where businesses are competitive, the insurer will lose some of its insureds and market share, receive less in premiums and pay fewer claims. In a hard market, the insurer will gain market share, receive more in premiums and pay more claims.

- **Mixed strategy**

  The insurer may partially follow the market premium loading. In doing so, the insurer may in a soft market incur lower underwriting losses than with the maintaining market share strategy and yet keep more insureds than with the conserving capital strategy.

Underwriting cycles can have a significant impact on the stability (ruin probability) of property/casualty insurance companies. However, there is little in the risk theory literature providing tools to quantify the additional risk associated with these cycles. Using simulation techniques, Daykin *et al.* (1994) study the relationship between underwriting cycles and ruin probabilities. Further results related to this topic may be found under the title of "dynamic financial analysis." See for example, D'Arcy *et al.* (1997) and Kaufmann *et al.* (2001).

In this paper, we explore an insurer surplus model that reflects the impact of underwriting cycles on insurers' ruin probabilities and we study the above strategies for coping with the cyclic business environment. We do not intend to analyze the cause of underwriting cycles. Rather, we focus on the question of how an insurance company can deal these cycles given that they occur.

The paper is organized as follows. In Section 2, we present an insurer surplus model that allows underwriting cycles, with parameters that reflect the magnitude of the cycles, insureds' sensitivity to the cycles, and the insurer's strategy for responding to the cycles. Ruin probabilities under the proposed model are discussed in Section 3. We first present some simulation results to gain an understanding of the behavior of the ruin probabilities and how they are affected by the various model parameters. We then derive a Lundberg-type upper bound for the ultimate ruin probability, and, in Section 4, we use the upper bound to explore the different strategies. This approach is justified by the results of a simulation study presented in Section 5. Some concluding remarks are provided in Section 6.

2 The Model

In the dynamic financial analysis literature, the underwriting cycle is considered by assuming that the underwriting environment shifts among soft and hard markets according to a Markov chain (see, for example, Kaufmann *et al.*, 2001, and D'Arcy *et al.*, 1998). In this paper, we model the underwriting cycle by assuming that the risk loading and the claim rate follow a deterministic cyclic function. This seems reasonable because, in practice, the claim rate varies continuously rather than jumping between levels. Further, as pointed out by Asmussen and Rolski (1994), as the number of market states increases, the Markovian model converges to a continuous, deterministic, periodic model.

2.1 Model requirements

In order to appropriately model an insurer's surplus when faced with underwriting cycles, we must first recognize that there is a relationship between the exposures in force and the risk-loaded premium per exposure, and that this relationship changes throughout the cycle. During a soft market, the insurer must charge less per exposure in order to retain the same number of exposures in force than it can charge during a hard market. Therefore, if the insurer fixes the premium per exposure, the exposures in force will vary cyclicly over time. If the insurer continually adjusts its premium in an effort to maintain a stable number of exposures in force, the the premium will vary cyclicly over time. It is then necessary for the surplus model to allow both the premium and the exposures to vary cyclicly. The former can be accommodated by defining the relative security loading to be a cyclic function, and the latter can be allowed for by defining the claim rate to be a cyclic function of time.

There is some literature on periodic risk processes. Asmussen and Rolski (1994) gave some important theoretic results related to the periodic risk model. Chukova *et al.* (2000) studied a periodic risk process from a reliability theory point of view. Morales (2004) provided a practical simulation methodology for a risk process with periodic claim intensity. Lu and Garrido (2005) studied estimation method of a risk process with periodic claim intensity.

In this paper, we allow a component of each of the relative security loading function and the claim rate function to be a trigonometric function. This not only produces smooth cyclic behavior, but the resulting functions are convenient mathematically. In our model, we assume a deterministic underwriting cycle of length 2π (about 6) years. Though this period is approximately that which has been observed, one could, without difficulty, assume a different period.

In most models of the surplus process, expenses are ignored. The rationale is that the expense loading which is added to the premiums will cover the expenses incurred, with minimal risk that this is not the case. When the exposures fluctuate over time, the risk is more significant. In particular, fixed expenses (e.g. overhead) are difficult to cover at all times when the expense loading is fluctuating. We reflect this additional risk in our model by subtracting the fixed expense rate from the loaded premium rate.

2.2 The surplus process

Let u denote the insurer's initial surplus and $U_s(t)$ denote the insurer's surplus at time t, where s is the initial state of the cycle, which we also refer to as the initial market status, $0 \leq s < 2\pi$. Assume that $U_s(t)$ is given by

$$U_s(t) = u + P_s(t) - \sum_{i=1}^{N_s(t)} X_i, \tag{1}$$

where $P_s(t)$ represents the cumulative premium income by time t, $N_s(t)$ is the number of claims by time t, and $X_1, X_2, \ldots$ are iid claim amount random variables with distribution function F, moment generation function $\hat{F}$ and mean μ. We assume that $\{N_s(t), t \geq 0\}$ is a time–inhomogeneous Poisson process with rate $\lambda_s(t)$ at time t. These assumptions imply that, while the claim rate varies over time to reflect the changing exposures, the risk profile of the insured group does not change. That is, the claim amount distribution does not depend on time.

Define

$$S_s(t) = u - U_s(t) = \sum_{i=1}^{N_s(t)} X_i - P_s(t)$$

to be the aggregate loss by time t. Then the probability of ruin is

$$\begin{aligned}\psi_s(u) &= \Pr\left(\inf_{t\geq 0} U_s(t) < 0\right)\\ &= \Pr\left(\sup_{t\geq 0} S_s(t) > u\right).\end{aligned}$$

2.3 The security loading

An insurer can adjust its premium per exposure by changing its security loading over time. We therefore assume that the loading is given by

$$\theta_s(t) = \theta + A\,c\,\sin(s+t), \tag{2}$$

where A reflects the magnitude of the underwriting cycle and $0 \leq c \leq 1$ is the insurer's strategy parameter. When $c = 0$, the insurer is using the conserving capital strategy. When $c = 1$, the insurer is using the maintaining market share strategy. When $0 < c < 1$, the insurer is employing a mixed strategy. The value of A then determines the amplitude of loading fluctuation needed to keep the number of exposure units constant over time. Figure 1 shows the graphs of the function $\theta_0(t)$ for three different values of the strategy parameter c. The parameter $A = 0.5$ was chosen to produce cycles with a rather large magnitude. Three cycles have been plotted. The first half of each cycle (0 to π, 2π to 3π, and 4π to 5π) represents a hard market. The second half of each cycle represents a soft market.

2.4 The claim rate

As described in Feldblum (1996), because insureds seek the best possible insurance price, an insurer's exposures, and therefore claim rate, are inversely related to its premium loading. To represent this insured turnover effect, we assume that the insurer's claim rate at time t is

$$\lambda_s(t) = \lambda \cdot (1 - A\,B(1-c)\cos(s+t)), \tag{3}$$

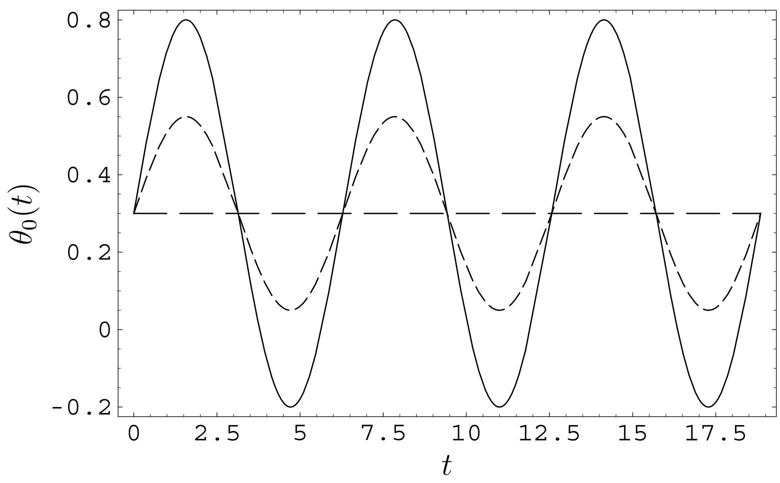

Figure 1: Plots of $\theta_0(t)$ for $c = 0$ (long dashed), $c = 0.5$ (short dashed), and $c = 1$ (solid), with $\theta = 0.3$ and $A = 0.5$.

where B represents the sensitivity of policyholders to departures of the insurer's loading from the loading that prevails in the market. In equation (3), $A\,B(1-c)$ represents the magnitude of claim rate fluctuations that occur when the insurer chooses strategy c. With $c = 1$, the insurer follows exactly the market price. Thus, no insured turnover occurs and the insurer's claim rate always remains at λ. With $c = 0$, the insurer is totally disregarding the market status. As a result, it loses insureds in a soft market (resulting in the claim rate decreasing to $\lambda \cdot (1 - A\,B)$ by the end of the soft market) and gains insureds in a hard market (resulting in the claim rate increasing to $\lambda \cdot (1 + A\,B)$ by the end of the hard market). With $0 < c < 1$, the insurer partially follows the market price. Its claim rate varies with the market cycle but to a lesser extent than with $c = 0$. Note that the parameter B can be no more than $1/A$. Otherwise the claim rate becomes negative when $c = 0$.

Figure 2 show the claim rate that results from different values of the strategy parameter, c. Again, the parameter $A = 0.5$, and the parameter B was chosen to be 1.8. This implies that policyholders are highly sensitive to premium differences.

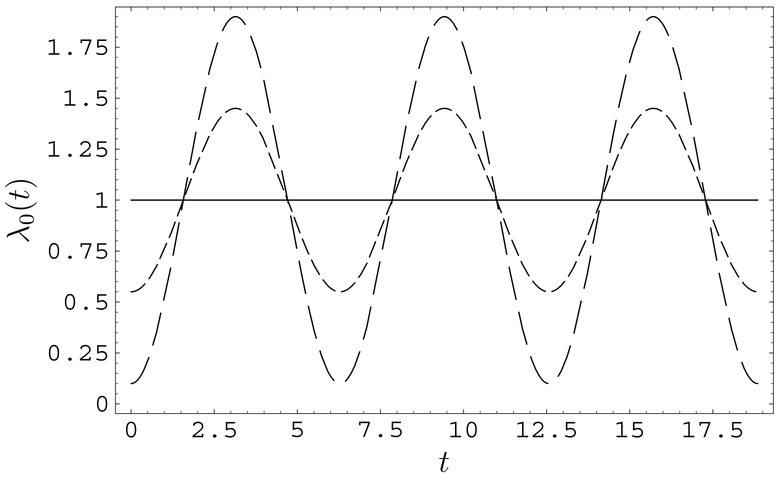

Figure 2: Plots of $\lambda_0(t)$ for $c = 0$ (long dashed), $c = 0.5$ (short dashed), and $c = 1$ (solid), with $\lambda = 1$, $A = 0.5$ and $B = 1.8$.

2.5 The net premium rate

Based on the above definitions of the security loading and the claim rate, the net premium rate at time t is given by

$$p_s(t) = (1 + \theta_s(t))\lambda_s(t)\mu - E, \tag{4}$$

where E is the fixed expense rate, and $\theta_s(t)$ and $\lambda_s(t)$ are given by equations (2) and (3), respectively. Since these functions both have a period of 2π, so does $p_s(t)$. However, we note that the premium rate reaches its maximum and minimum at different times during the cycle for different values of c. Figure 3 shows plots of $p_0(t)$ for different values of c based on the $\theta_0(t)$ and $\lambda_0(t)$ plotted in Figures 1 and 2, respectively, and $E = 0.1$.

Integrating the right hand side of equation (4) from zero to t yields the cumulative net premium at time t. That is,

$$\begin{aligned} P_s(t) &= \int_0^t (1 + \theta + Ac\sin(s+t))(\lambda\mu(1 - B(1-c)\cos(s+t)))dt - Et \\ &= ((1+\theta)\lambda\mu - E)t + Ac\lambda\mu(\cos(s) - \cos(s+t)) \\ &\qquad - A\,B(1-c)\lambda\mu(1+\theta)(\sin(s+t) - \sin(s)) \\ &\qquad\quad - A^2Bc(1-c)\lambda\mu(\sin^2(s+t) - \sin^2(s)). \end{aligned} \tag{5}$$

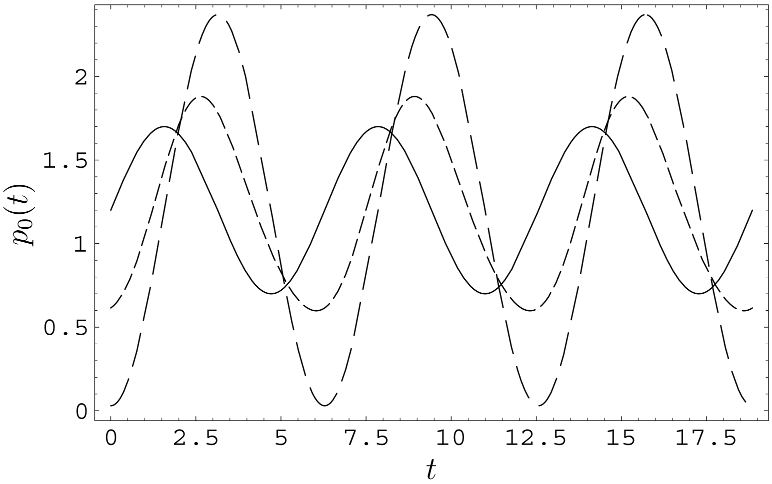

Figure 3: Plots of $p_0(t)$ for $c = 0$ (long dashed), $c = 0.5$ (short dashed), and $c = 1$ (solid), with $\theta = 0.3$, $\lambda = 1$, $\mu = 1$, $A = 0.5$, $B = 1.8$, and $E = 0.1$.

Three remarks are appropriate at this point:

1. The cumulative premium increases by a constant amount per period. That is,

$$\begin{aligned} P_s(t + 2\pi) &= 2\pi((1 + \theta)\lambda\mu - E) + P_s(t) \\ &= P_s(2\pi) + P_s(t). \end{aligned}$$

   This will be useful in our later calculations.

2. The average claim rate is given by

$$\lambda^* = \frac{1}{2\pi}\int_0^{2\pi} \lambda_s(t)dt = \lambda, \tag{6}$$

   and the average net premium rate is given by

$$p^* = \frac{1}{2\pi}\int_0^{2\pi} p_s(t)dt = (1 + \theta)\lambda\mu - E. \tag{7}$$

3. The average net profit loading is $\theta - \frac{E}{\lambda\mu}$, and this quantity must be positive. Otherwise, ruin is certain. Since the average net profit loading is independent of the strategy c, it is fair to compare the ruin probability under different strategies, which we do in Section 3.

2.6 The expected surplus

From equation (1), it is easily seen that the expected surplus at time t is given by

$$\mathbb{E}[U_s(t)] = u + P_s(t) - \Lambda_s(t)\mu, \tag{8}$$

where $\Lambda_s(t) = \int_0^t \lambda_s(u)du$. While the expected surplus tells us nothing about the variability of the surplus at a given point in time, it helps us to understand how the process behaves on average. It is interesting to observe the expected surplus for different values of the strategy parameter. Plots of $\mathbb{E}[U_0(t)]$ are shown in Figure 4 for the same parameter values as in Figures 1 to 3. Figure 4 shows that, at most points in time, the expected surplus is smallest for $c = 0$ and largest for $c = 1$. This is largely due to the parameter values chosen and, in particular, the initial state of the cycle which is $s = 0$. Notice from Figure 3 that the early premiums are lowest when $c = 0$ and highest when $c = 1$, though the total premium income in a cycle is independent of the strategy. Figure 5 shows plots of $\mathbb{E}[U_\pi(t)]$ corresponding to those of $\mathbb{E}[U_0(t)]$ shown in Figure 4. We see that when $s = \pi$, the expected surplus is, at most points in time, smallest for $c = 1$ and largest for $c = 0$.

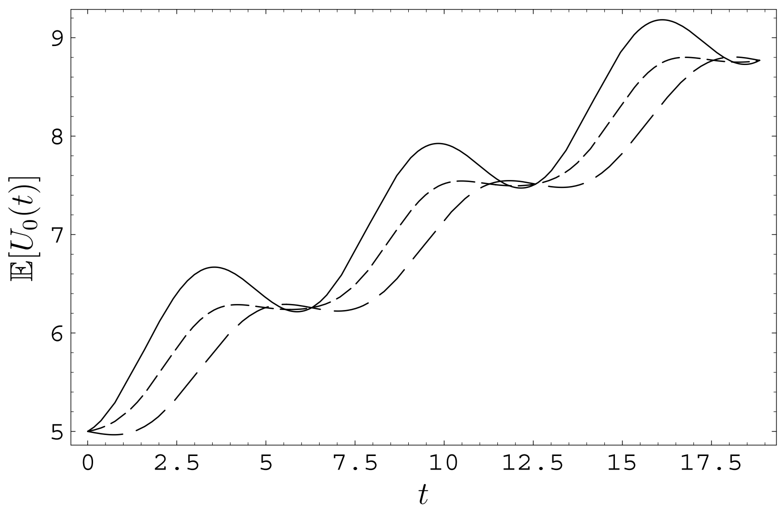

Figure 4: Plots of $\mathbb{E}[U_0(t)]$ for $c = 0$ (long dashed), $c = 0.5$ (short dashed), and $c = 1$ (solid), with $\theta = 0.3$, $\lambda = 1$, $\mu = 1$, $A = 0.5$, $B = 1.8$, $E = 0.1$, and $u = 5$.

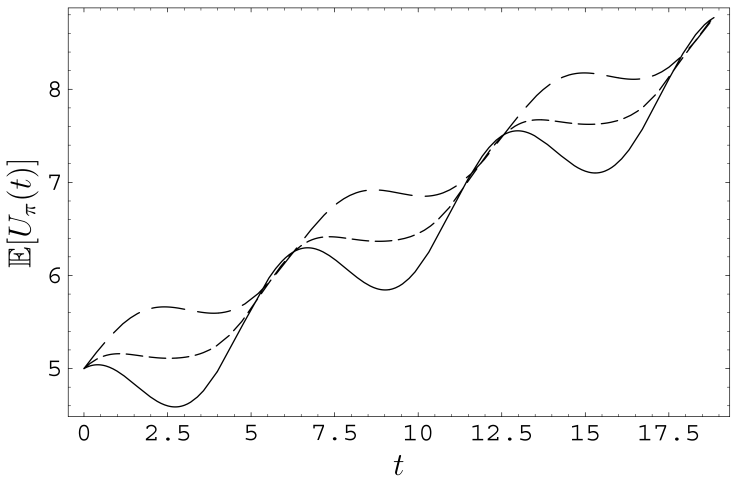

Figure 5: Plots of $\mathbb{E}[U_\pi(t)]$ for $c = 0$ (long dashed), $c = 0.5$ (short dashed), and $c = 1$ (solid), with $\theta = 0.3$, $\lambda = 1$, $\mu = 1$, $A = 0.5$, $B = 1.8$, $E = 0.1$, and $u = 5$.

We might anticipate that ruin probabilities will be higher when the expected surplus is lower, since at most point in time there is a higher probability that a claim will cause ruin. This insight helps us to interpret the results of the next section.

3 Ruin Probabilities

The model presented in Section 2 is considerably more complicated than the classical risk model. It is therefore more difficult to to explore ruin probabilities analytically. We shall consider the development of Lundberg-type upper bounds for the ultimate ruin probability. However, we first examine some ruin probability estimates obtained by simulation. This will provide an understanding of how the ruin probabilities behave and how they depend on the model parameters.

3.1 Simulation Results

Estimates of the probability of ultimate ruin with initial surplus 5 are shown for different values of A, B, c, and s in Table 1. Each estimate was obtained by simulating the surplus process 100,000 times over a time horizon of 100 years assuming that claim amounts have a standard exponential distribution. Some testing showed that the probability of ruin after 100 years is negligible, and therefore a 100 year horizon is suitable for estimating the probability of ultimate ruin. The results obtained for $A = 0.5$ and $B = 1.8$ show most dramatically the differences in the ruin probabilities for different values of c and s. This is to be expected since, for these A and B, the magnitude of each cycle is large, and policyholders are highly sensitive to the cycles.

We observe that, if the cycle is just entering a hard market ($s = 0$) at time 0, then the ruin probability decreases with increasing c and is smallest when $c = 1$. This is to be expected since the premium income is higher during a hard market if the company adopts the maintaining market share ($c = 1$) strategy. However, if the cycle is just entering a soft market ($s = \pi$) at time 0, then the ruin probability increases with increasing c and is smallest when $c = 0$. For $s = \pi/2$ or $3\pi/2$, it is less clear which strategy leads to the smallest ruin probability, though it appears that a mixed strategy is best ($0 < c < 1$).

Table 2 shows the corresponding ruin probability estimates when the time horizon is 10 years. Though the estimates are all smaller than the ultimate ruin probability estimates, we observe similar relationships to those in Table 1. To gain a better understanding of how the ruin probability depends on the strategy parameter, c, for different values of the initial state parameter, s, we performed simulations for a larger number of c values for the case in which $A = 0.5$, $B = 1.8$, and the time horizon is 10 years. The results are shown in Table 3. Despite the variability due to randomness, the table reveals roughly how the the optimal strategy parameter (the value of c that produces the lowest ruin probability) varies with the initial state of the cycle. This is explored further in Section 4.

3.2 Lundberg upper bound for the ruin probability

Since analytical solution for the ruin probability is untractable, in this section, we compare the Lundberg upper bound of ruin probabilities under different strategies.

As in Asmussen and Rolski (1994), let γ^* be the the adjustment coefficient for the average risk process with constant premium rate p^* and claim rate λ^* given by equations (7) and (6). Then

$$\int_0^{2\pi} \lambda_s(v)dv[\hat{F}(\gamma^*) - 1] = \gamma^* P_s(2\pi), \tag{9}$$

or more explicitly,

$$\lambda[\hat{F}(\gamma^*) - 1] = \gamma^*((1+\theta)\lambda\mu - E). \tag{10}$$

We now present our main theoretical result.

Theorem 3.1 *For the surplus process with initial market status s, the ultimate probability of ruin is given by*

$$\psi_s(u) \quad \leq \quad e^{-\gamma^* u} \cdot e^{\gamma^* h_s(c)}, \tag{11}$$

where

$$\begin{aligned} h_s(c) \quad = \quad - \inf_{0 \leq t \leq 2\pi} \Big\{ & Ac\lambda\mu(\cos(s) - \cos(s+t)) - AB(1-c)E(\sin(s) - \sin(s+t)) \\ & - \frac{1}{2}A^2Bc(1-c)\lambda\mu(\sin^2(s) - \sin^2(s+t)) \Big\}. \end{aligned} \tag{12}$$

Proof. See appendix.

As a function of initial market status, s, and an individual insurer's control, c, $h_s(c)$ is positively related to the upper bound of ruin probability and may be viewed as a measure of how the initial market status and the strategy may affect the insurer's ruin probability, as will be shown.

We remark that since γ^* is the adjustment coefficient of the average risk process, $e^{-\gamma^* u}$ gives an upper bound for the ruin probability for this process.

4 Comparison of strategies

4.1 Maintaining Market Share

With the maintaining market share strategy, $c = 1$,

$$h_s(c) \quad = \quad A\lambda\mu(1 - \cos(s)). \tag{13}$$

Therefore, the probability of ruin

$$\psi_{(s)}(u) \le e^{-\gamma^* u} e^{\gamma^* A\lambda\mu(1-\cos(s))}. \tag{14}$$

It is clear from equation (13) that with this strategy, the ruin probability is positively related to the magnitude of the underwriting cycle.

We remark here that, with $s = 0$,

$$P_0(t) = ((1+\theta)\lambda\mu - E)t + A\lambda\mu(1 - \cos(t)) \ge ((1+\theta)\lambda\mu - E)t$$

for all t. This means that, in this case, the insurer always accumulates more premiums than if it were collecting premiums at a constant average rate. Therefore, its ruin probability must be less than or equal to that of the average process. However, our upper bound is exactly the same as that of the average process.

4.2 Conserving capital strategy

With the conserving capital strategy, $c = 0$,

$$h_s(c) = ABE(1 - \sin(s)). \tag{15}$$

Thus, the probability of ruin

$$\psi^{(s)}(u) \le e^{-\gamma^* u} e^{\gamma^* ABE(1-\sin(s))}. \tag{16}$$

It is clear from equation (15) that with this strategy, the ruin probability is positively related to the magnitude of the underwriting cycle, A, the policyholder sensitivity, B, and fixed expense rate E.

4.3 Optimal Strategy

To compare the two strategies, we plot the ruin probability upper bound $e^{-\gamma^* u} \cdot e^{\gamma^* h_s(c)}$ for the maintaining market share strategy and for the conserving capital strategy on the same

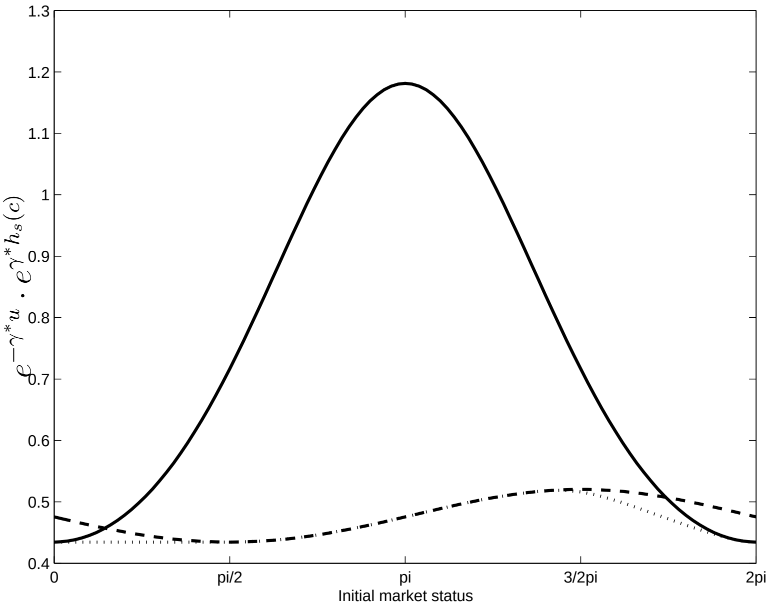

Figure 6: Plots of the ruin probability upper bounds for the maintaining market share (solid), conserving capital (dashed) and optimal (dotted) strategies, with $\theta = 0.3$, $\lambda = 1$, $\mu = 1$, $A = 0.5$, $B = 1.8$, $E = 0.1$, and $u = 5$.

scale in Figure 6. The graphs are based on the parameter values $\theta = 0.3$, $\lambda = 1$, $\mu = 1$, $A = 0.5$, $B = 1.8$, $E = 0.1$, and $u = 5$.

Figure 6 indicates that when s is close to the end of the cycle (0 or 2π), the maintaining market share strategy has a lower ruin probability upper bound, and when s is near the middle of the cycle, the conserving capital strategy has a lower ruin probability upper bound.

Furthermore, for each initial market status, s, an "optimal" strategy, c, may be selected such that $h_s(c)$ and thus the upper bound of ruin probability is minimized. Here we use a numerical routine to find the optimal strategy for each s and plot it in Figure 7. The upper bound corresponding to this strategy is shown as the dotted line in Figure 6.

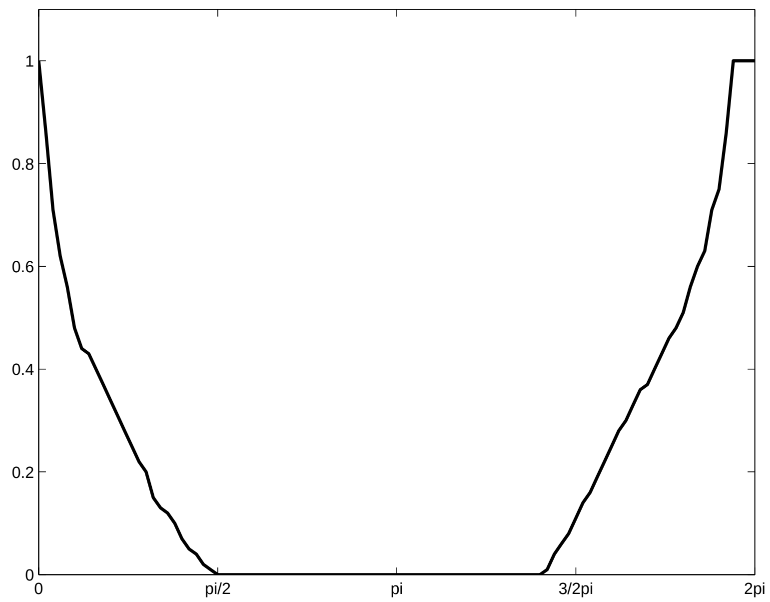

Figure 7: Plot of the optimal strategy parameter value, c, for different values of the initial market status s, with $\theta = 0.3$, $\lambda = 1$, $\mu = 1$, $A = 0.5$, $B = 1.8$, $E = 0.1$, and $u = 5$.

Figure 7 suggests that in order to minimize the ruin probability, the insurer might gradually move from a maintaining market share strategy ($c = 1$) to a conserving capital strategy ($c = 0$) as the market moves from its equilibrium to its zenith. The insurer should continue this strategy while the market moves from its zenith to its nadir. Then the insurer should gradually move back to a maintaining market share strategy when the underwriting cycle moves from its nadir back to equilibrium.

From Figure 6, we see that the ruin probability upper bound is uniformly lower under the optimal strategy than with both the maintaining market share and the conserving capital strategies. Furthermore, we see that, for $0 < s < \pi/2$ when market prices move from the equilibrium point to the zenith, we may select c such that $h_s(c) = 0$ and the upper bound for the ruin probability remains the same as the average model. This was pointed out in subsection 4.1.

One may notice that the upper bound given by Theorem 3.1 may exceed one in some cases and thus may not be a good approximation to the ruin probability. We contend that it is the comparative value that matters. That is, a higher upper bound indicates higher risk and vice-versa. Therefore, the bound is very useful in comparing strategies.

5 How good is the upper bound?

In this section, we examine whether the ruin probability bounds obtained in this paper give good indications of the behavior of the true ruin probability for different values of s, the initial market status. In particular, we obtain the approximate ultimate ruin probabilities by straightforward simulation and compare them with the upper bounds.

In this simulation, we assume that the initial surplus is 5 and claims follow an exponential distribution with mean $\mu = 1$. We use the parameters $A = 0.5$, $B = 1.8$, $\lambda = 1$, $\theta = 0.3$, and $E = 0.1$. With these parameters, the adjustment coefficient for the average process is $\gamma^* = 1/6$. Thus, the Lundberg upper bound for the ruin probability of the average process is $e^{-\gamma^* u} = e^{-5/6} = 0.434598$, and the exact ruin probability for the average process is $\frac{1}{1+\theta-E} e^{-\gamma^* u} = 0.362165$.

The simulated ultimate ruin probabilities for the risk processes with insurer's control $c = 1$, $c = 0$ and the optimal control c are plotted in Figure 8, and the optimal values of c based on the simulated ruin probabilities are plotted in Figure 9. Note that the simulated ruin probabilities underestimate the true ruin probabilities because a 100 year (rather then

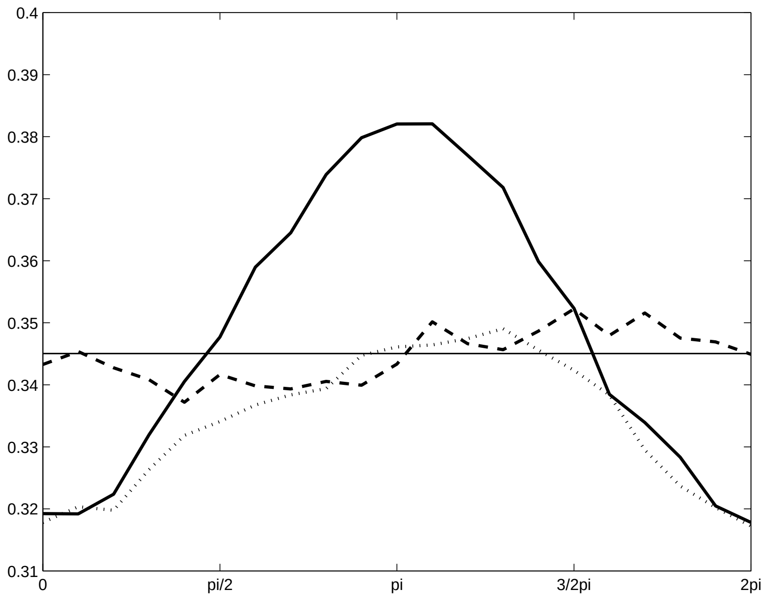

Figure 8: Plots of simulated ruin probabilities for the maintaining market share (solid), conserving capital (dashed) and optimal (dotted) strategies, with $\theta = 0.3$, $\lambda = 1$, $\mu = 1$, $A = 0.5$, $B = 1.8$, $E = 0.1$, and $u = 5$.

infinite) time horizon was used in performing the simulations.

The pattern in Figures 6 and 8 are quite similar. Therefore, we may claim that the upper bounds derived here are good indicators of the behavior of the ultimate ruin probabilities. Consequently, the methods provided in Sections 3 and 4 may be useful for analyzing the ruin probabilities of insurers utilizing different strategies to cope with the underwriting cycle.

6 Summary and Conclusions

We have presented a surplus model that reflects the impact of underwriting cycles on the insurer's surplus. The model includes a strategy parameter that indicates how the insurer responds to the cycles. The model allows one to analyze ruin probabilities under the different strategies.

Though the complexity of the models limits our ability to obtain analytical results for

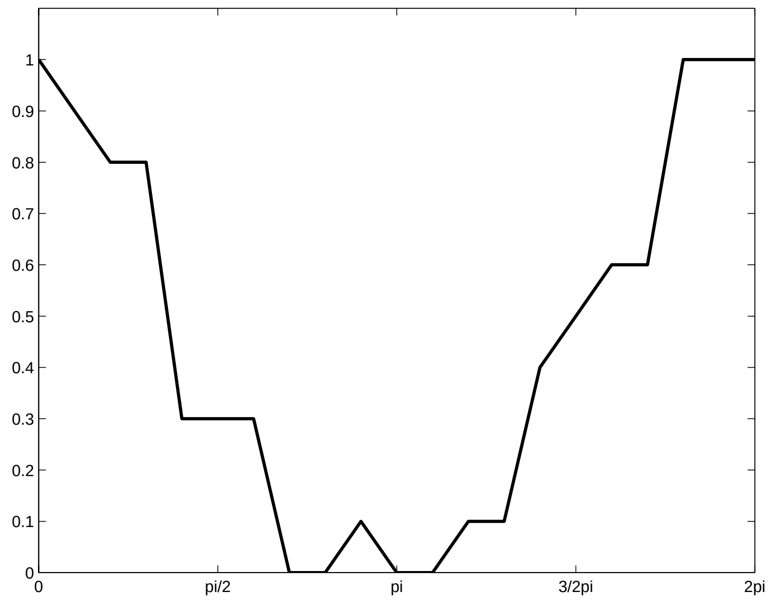

Figure 9: Plot of the optimal strategy parameter value, c, for different values of the initial market status s, with $\theta = 0.3$, $\lambda = 1$, $\mu = 1$, $A = 0.5$, $B = 1.8$, $E = 0.1$, and $u = 5$.

the ruin probabilities, we have presented a Lundberg-type upper bound for the ultimate ruin probability and showed using simulation that it behaves similarly to the actual probability. The upper bound is therefore useful in comparing the different strategies.

We note that our model is a static one, which assumes that the insurer does not change strategy c with the evolution of the underwriting cycle. If the insurer does change c over the cycle, a dynamic model must be used to analyze the ruin probability. We will leave this as a future research topic.

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A Appendix

Proof of Theorem 3.1. We begin by noting that $\mathbb{E}[e^{\alpha S_s(t)}]$ satisfies

$$\mathbb{E}[e^{\alpha S_s(t)}] \quad = \quad \exp\left(-\alpha P_s(t) + \int_0^t \lambda_s(v)dv[\hat{F}(\alpha) - 1]\right). \tag{17}$$

This result is similar to equation (3.1) in Asmussen and Rolski (1994), and we refer readers to their paper. Next we show that for α such that $\mathbb{E}[e^{\alpha S_s(t)}]$ exists and is finite,

$$M_\alpha(t) = \exp\left(\alpha S_s(t) + \alpha P_s(t) - \int_0^t \lambda_s(v)dv[\hat{F}(\alpha) - 1]\right) \tag{18}$$

is a $\mathcal{F}_t$-martingale, where $\mathcal{F}_t$ is the natural filtration on the Skorohod space of sample paths of $S_s(t)$. We have

$$\begin{aligned}
\mathbb{E}[M_\alpha(t+x)|\mathcal{F}_t] \quad &= \quad \mathbb{E}\left[\exp\left(\alpha S_s(t+x) + \alpha P_s(t+x) - \int_0^{t+x} \lambda_s(v)dv[\hat{F}(\alpha) - 1]\right)\middle|\mathcal{F}_t\right] \\
&= \quad M_\alpha(t) \cdot \mathbb{E}\Bigg[\exp\Bigg(\alpha(S_s(t+x) - S_s(t)) + \alpha(P_s(t+x) - P_s(t)) \\
&\qquad\qquad - \int_t^{t+x} \lambda_s(v)dv[\hat{F}(\alpha) - 1]\Bigg)\Bigg|\mathcal{F}_t\Bigg] \\
&= \quad M_\alpha(t) \cdot \mathbb{E}\Bigg[\exp\Bigg(\alpha(S_{s+t}(x)) + \alpha(P_{s+t}(x)) \\
&\qquad\qquad - \int_0^{x} \lambda_s(v+t)dv[\hat{F}(\alpha) - 1]\Bigg)\Bigg].
\end{aligned} \tag{19}$$

However, from (17),

$$\mathbb{E}[e^{\alpha S_{s+t}(x)}] \quad = \quad \exp\left(-\alpha P_{s+t}(x) + \int_0^x \lambda_{s+t}(v)dv[\hat{F}(\alpha) - 1]\right), \tag{20}$$

and since $\lambda_s(v+t) = \lambda_{s+t}(v)$ for every s, t and v, the exponent in equation (19) becomes zero and we have $\mathbb{E}[M_\alpha(t+x)|\mathcal{F}_t] = M_\alpha(t)$.

Inspired by Asmussen and Rolski (1994), we select $\alpha = \gamma^*$ in equation (18) and let $M(t)$ denote $M_{\gamma^*}(t)$. Then by equation (9) and the periodic properties of $\lambda_s(\cdot)$ and $P_s(\cdot)$, we have

$$\gamma^* P_s(2k\pi) = \int_0^{2k\pi} \lambda_s(v)dv[\hat{F}(\gamma^*) - 1],\ k = 0, 1, 2, \ldots. \tag{21}$$

Now let

$$\eta(t) = t - \left\lfloor \frac{t}{2\pi} \right\rfloor \cdot 2\pi,$$

where $\lfloor \cdot \rfloor$ denotes the greatest integer function. Clearly, $t - \eta(t)$ is an integer multiple of 2π, and $0 \le \eta(t) < 2\pi$. Then

$$\begin{aligned}
&\gamma^* P_s(t) - \int_0^t \lambda_s(v)dv[\hat{F}(\gamma^*) - 1] \\
&= \gamma^* P_s(t - \eta(t)) - \int_0^{t-\eta(t)} \lambda_s(v)dv[\hat{F}(\gamma^*) - 1] \\
&\qquad + \gamma^* P_{s+t-\eta(t)}(\eta(t)) - \int_{t-\eta(t)}^{t} \lambda_s(v)dv[\hat{F}(\gamma^*) - 1] \\
&= \gamma^* P_s(\eta(t)) - \int_0^{\eta(t)} \lambda_s(v)dv[\hat{F}(\gamma^*) - 1]. \qquad (22)
\end{aligned}$$

The last step follows from (21) using the fact that $t - \eta(t)$ is a multiple of 2π and by recognizing that $P_s(t)$ is a cyclic function of s and $\lambda_s(t)$ is a cyclic function of t, both with period 2π. Inserting (22) into equation (18) and taking expectations on both sides we obtain

$$\mathbb{E}[M(2k\pi)] = \mathbb{E}[e^{\gamma^* S_s(2k\pi)}],\ k = 0, 1, 2, \ldots.$$

Then, for any $t > 0$, we have

$$\begin{aligned}
\mathbb{E}[M(t)] &= \mathbb{E}[e^{\gamma^* S_s(t)}] \cdot \exp\left(\gamma^* P_s(\eta(t)) - \int_0^{\eta(t)} \lambda_s(v)dv[\hat{F}(\gamma^*) - 1]\right) \\
&= \mathbb{E}[e^{\gamma^* S_s(t)}] \cdot \\
&\qquad \exp\left(\gamma^* P_s(\eta(t)) - \gamma^*((1+\theta)\lambda\mu - E)\int_0^{\eta(t)} (1 - B(1-c)\cos(s+v))dv\right) \\
&\qquad\qquad \text{from (3) and (10)} \\
&= \mathbb{E}[e^{\gamma^* S_s(t)}] \cdot \\
&\qquad \exp\Big(\gamma^*\big(Ac\lambda\mu(\cos(s) - \cos(s+\eta(t))) - AB(1-c)E(\sin(s) - \sin(s+\eta(t))) \\
&\qquad\quad - \frac{1}{2}A^2Bc(1-c)\lambda\mu(\sin^2(s) - \sin^2(s+\eta(t)))\big)\Big) \\
&= \mathbb{E}[e^{\gamma^* S_s(t)}] \cdot e^{\gamma^* \chi_s(t)}, \qquad (23)
\end{aligned}$$

where

$$\begin{aligned}\chi_s(t) &= Ac\lambda\mu(\cos(s)-\cos(s+\eta(t))) - AB(1-c)E(\sin(s)-\sin(s+\eta(t))) \\ &\quad - \frac{1}{2}A^2Bc(1-c)\lambda\mu(\sin^2(s)-\sin^2(s+\eta(t))). \qquad (24)\end{aligned}$$

Let $\tau = \inf\{t: \; S(t) > u\}$ be the time of ruin, and let T be a constant. Then $\tau \wedge T = \min(\tau, T)$ is a bounded stopping time. A standard way of finding upper bounds for $\psi_{(s)}(u)$ is to apply the optional sampling theorem to $\tau \wedge T$ and then let T approach positive infinity. By the optional sampling theorem, we have

$$\mathbb{E}[M(\tau \wedge T)] = M(0) = 1. \qquad (25)$$

However,

$$\mathbb{E}[M(\tau \wedge T)] = \mathbb{E}[M(\tau)I(\tau \leq T)] + \mathbb{E}[M(T)I(\tau > T)], \qquad (26)$$

where $I(\cdot)$ is the indicator function. Since we assume a positive average security loading, $S_s(t) \to -\infty$ almost surely as $T \to \infty$, and we have

$$\begin{aligned}\mathbb{E}[M(T)I(\tau > T)] &\leq \mathbb{E}[M(T)] \\ &\leq \mathbb{E}[e^{\gamma^* S_s(T)}] \cdot e^{\gamma^* \max_{t\geq 0} \chi_s(t)} \\ &\to 0 \text{ as } T \to \infty.\end{aligned}$$

Therefore, letting $T \to \infty$ in (26) and using (25), we have

$$\mathbb{E}[M(\tau)I(\tau < \infty)] = 1. \qquad (27)$$

Then since

$$\begin{aligned}\mathbb{E}[M(\tau)I(\tau < \infty)] &= \mathbb{E}[M(\tau)|\tau < \infty]\,\mathrm{Pr}(\tau < \infty) \\ &= \mathbb{E}[M(\tau)|\tau < \infty]\psi_s(u),\end{aligned}$$

the ruin probability is given by

$$\psi_s(u) = \frac{1}{\mathbb{E}[e^{\gamma^* S(\tau) + \gamma^* \chi_s(\tau)}|\tau < \infty]}. \qquad (28)$$

Let $S(\tau) = u + \xi(u)$ with $\xi(u) > 0$ being the deficit at ruin. Then

$$\begin{aligned}
\psi_s(u) &= \frac{e^{-\gamma^* u}}{\mathbb{E}[e^{\gamma^* \xi(u) + \gamma^* \chi_s(\tau)}]} \\
&\leq \frac{e^{-\gamma^* u}}{\mathbb{E}[e^{\gamma^* \chi_s(\tau)}]} \\
&\leq e^{-\gamma^* u} \cdot e^{-\gamma^* \inf\limits_{0 \leq t \leq 2\pi} (\chi_s(t))} \\
&= e^{-\gamma^* u} \cdot e^{\gamma^* h_s(c)}
\end{aligned} \tag{29}$$

The third step is due to the fact that for $0 \leq t < 2\pi$, $\eta(t) = t$. This completes the proof. ■

Table 1: Estimates of the Probability of Ultimate Ruin with $\theta = 0.3$, $\lambda = 1$, $\mu = 1$, $E = 0.1$, and $u = 5$. 100,000 simulations of the surplus process over a time horizon of 100 years were used to obtain each estimate.

Results for $A = 0.5$ and $B = 1.8$									
	Initial State of Cycle, s								
c	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π
0	0.3459	0.3424	0.3386	0.3446	0.3455	0.3469	0.3488	0.3519	0.3455
0.5	0.3295	0.3299	0.3370	0.3506	0.3595	0.3561	0.3448	0.3337	0.3289
1	0.3189	0.3281	0.3474	0.3718	0.3835	0.3751	0.3498	0.3269	0.3196
Results for $A = 0.5$ and $B = 0.5$									
	Initial State of Cycle, s								
c	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π
0	0.3444	0.3431	0.3431	0.3443	0.3441	0.3465	0.3472	0.3446	0.3452
0.5	0.3302	0.3317	0.3403	0.3551	0.3641	0.3596	0.3446	0.3381	0.3281
1	0.3175	0.3265	0.3443	0.3717	0.3839	0.3727	0.3518	0.3276	0.3182
Results for $A = 0.1$ and $B = 1.8$									
	Initial State of Cycle, s								
c	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π
0	0.3467	0.3444	0.3457	0.3453	0.3461	0.3489	0.3465	0.3444	0.3434
0.5	0.3400	0.3405	0.3442	0.3458	0.3457	0.3469	0.3446	0.3445	0.3414
1	0.3406	0.3416	0.3441	0.3518	0.3509	0.3488	0.3463	0.3388	0.3390
Results for $A = 0.1$ and $B = 0.5$									
	Initial State of Cycle, s								
c	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π
0	0.3449	0.3446	0.3439	0.3487	0.3438	0.3441	0.3474	0.3440	0.3454
0.5	0.3416	0.3445	0.3424	0.3466	0.3478	0.3452	0.3453	0.3416	0.3428
1	0.3372	0.3426	0.3449	0.3502	0.3524	0.3484	0.3447	0.3399	0.3413

Table 2: Estimates of the Probability of Ruin within 10 Years with $\theta = 0.3$, $\lambda = 1$, $\mu = 1$, $E = 0.1$, and $u = 5$. 100,000 simulations of the surplus process were used to obtain each estimate.

Results for $A = 0.5$ and $B = 1.8$									
	Initial State of Cycle, s								
c	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π
0	0.1681	0.1749	0.1726	0.1652	0.1542	0.1497	0.1517	0.1591	0.1702
0.5	0.1497	0.1562	0.1631	0.1727	0.1767	0.1668	0.1564	0.1510	0.1516
1	0.1378	0.1452	0.1640	0.1860	0.2020	0.1915	0.1704	0.1455	0.1357
Results for $A = 0.5$ and $B = 0.5$									
	Initial State of Cycle, s								
c	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π
0	0.1672	0.1683	0.1649	0.1646	0.1623	0.1613	0.1619	0.1591	0.1669
0.5	0.1505	0.1538	0.1618	0.1751	0.1773	0.1722	0.1609	0.1511	0.1472
1	0.1362	0.1416	0.1615	0.1865	0.2010	0.1915	0.1675	0.1469	0.1348
Results for $A = 0.1$ and $B = 1.8$									
	Initial State of Cycle, s								
c	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π
0	0.1643	0.1665	0.1653	0.1628	0.1635	0.1616	0.1622	0.1638	0.1652
0.5	0.1608	0.1625	0.1628	0.1639	0.1682	0.1674	0.1634	0.1609	0.1612
1	0.1553	0.1566	0.1606	0.1651	0.1697	0.1641	0.1626	0.1581	0.1567
Results for $A = 0.1$ and $B = 0.5$									
	Initial State of Cycle, s								
c	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π
0	0.1643	0.1647	0.1614	0.1621	0.1615	0.16214	0.1617	0.1626	0.1642
0.5	0.1608	0.1597	0.1635	0.1638	0.1675	0.1654	0.1619	0.1590	0.1599
1	0.1556	0.1594	0.1641	0.1663	0.1684	0.1681	0.1650	0.1563	0.1558

Table 3: Estimates of the Probability of Ruin within 10 Years with $\theta = 0.3$, $\lambda = 1$, $\mu = 1$, $E = 0.1$, and $u = 5$. 100,000 simulations of the surplus process were used to obtain each estimate. The smallest estimate in each column is shown in bold

Results for $A = 0.5$ and $B = 1.8$									
	Initial State of Cycle, s								
c	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π
0	0.1701	0.1768	0.1701	**0.1653**	**0.1548**	**0.1496**	0.1501	0.1586	0.1683
0.1	0.1666	0.1691	0.1701	0.1674	0.1593	0.1517	**0.1493**	0.1574	0.1643
0.2	0.1621	0.1674	0.1677	0.1659	0.1635	0.1567	0.1523	0.1540	0.1604
0.3	0.1584	0.1641	0.1647	0.1690	0.1655	0.1585	0.1511	0.1529	0.1581
0.4	0.1549	0.1609	0.1636	0.1687	0.1736	0.1637	0.1531	0.1527	0.1566
0.5	0.1531	0.1562	0.1624	0.1713	0.1785	0.1697	0.1551	0.1522	0.1501
0.6	0.1467	0.1535	0.1639	0.1758	0.1815	0.1721	0.1578	0.1489	0.1470
0.7	0.1453	0.1499	0.1614	0.1767	0.1856	0.1771	0.1597	0.1491	0.1436
0.8	0.1403	0.1460	**0.1612**	0.1828	0.1886	0.1825	0.1609	0.1493	0.1410
0.9	0.1395	0.1458	0.1634	0.1856	0.1971	0.1865	0.1653	**0.1445**	0.1396
1.0	**0.1367**	**0.1427**	0.1641	0.1885	0.2012	0.1926	0.1656	0.1486	**0.1356**