Fuzzy Post-Retirement Solvency Concepts: Some Preliminary Observations

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Abstract

This article presents some preliminary observations from a study that investigates and models the fuzziness inherent in post-retirement financial strategies. The focus here is on how fuzzy post-retirement solvency concepts can be represented, and the goal is to give the reader a flavor of the issues involved.

Key words: conservativeness, fuzzy, post-retirement, preservation of principal, retirement, sense of security, solvency, sufficiency

1. Introduction

Many of the concepts associated with post-retirement financial strategies[1] are fuzzy. From the perspective of the individual, descriptive terms like adequate, suitable and best fall into this category. From the perspective of the analysis upon which retirement decisions are made, the sources of fuzziness include modeling choices, parameter choices, inference processes, and boundary conditions. These fuzzy concepts impede the communication between the individual and his or her financial advisor, and limit the suitability of quantitative analysis from which a post-retirement strategy for an individual can be developed.

Three questions come to mind when confronted with these fuzzy post-retirement concepts: (1) how can these concepts be represented, (2) how can these concepts be coordinated, and (3) how can these concepts be implemented to formulate a tailor-made post-retirement strategy for an individual? This article, which presents some preliminary observations from a study that investigates and models the responses to these three questions, deals mainly with the first question. Its focus is on fuzzy post-retirement solvency issues. The goal is to present the reader with a flavor of the issues involved, and, to this end, the discussion generally is in conceptual rather than technical terms.

[1] Post-retirement financial strategies are discussed in Shapiro (2010).

2. Fuzzy Solvency Issues

This section presents some observations as they relate to fuzzy post-retirement solvency issues. The topics addressed are conservativeness, sufficiency, the relationship between retirement funds and a sense of security, and the preservation of principal.

2.1 Conservativeness

Following Shapiro (1990), we define methodologies and assumptions to be conservative (C) if they tend to produce gains. Figure 1 shows how a membership function (MF) for the notion of low conservativeness (C_L) might be represented, based on the metric relative gain

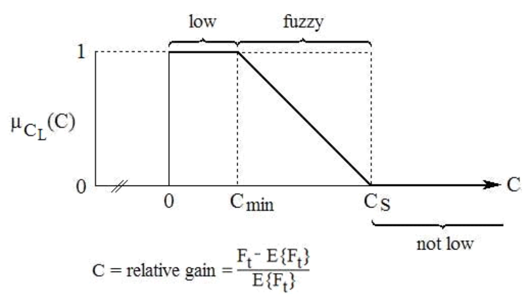

Figure 1: Low Conservativeness as a MF

Such a set is characterized by a MF, $\mu_{C_L}(C)$ here, which takes the values

$$\mu_{C_L}(C) = \begin{cases} 0 & C < 0 \\ 1 & 0 \le C < C_{min} \\ \frac{C_S - C}{C_S - C_{min}} & C_{min} \le C < C_S \\ 0 & C \ge C_S \end{cases} \quad (1)$$

As indicated, $\mu_{C_L}(C)$ assigns to each object a grade of membership (GOM) ranging between zero and one. Low conservativeness, when between zero and C_{min}, the minimum desired value, is assigned a grade of one. Relative gains of C_S, or more, which are satisfactory, are assigned a GOM of zero, that is, defined as not at all a member of the low conservativeness group. Between those values, (C_{min}, C_S), the GOM is fuzzy. If the MF has the shape depicted in Figure 1, it is characterized as reverse-S-shaped.

Note that at the low side, if the relative gains are negative, that is, if losses occur, that clearly is not conservative and so the MF for conservativeness is cropped at zero.

2.1.1 The probability of low conservativeness

As a simple example of how fuzziness may be implemented, consider the fuzzy risk of a low conservativeness position.[2] Let $\tilde{P}(C_L)$ denotes the probability of the fuzzy event low conservativeness. Then, in general,

$$\begin{aligned}\tilde{P}(C_L) &= \int_{\mathbb{R}^N} \mu_{C_L}(C)\,dP \\ &= \int_{\mathbb{R}^N} \mu_{C_L}(C)\,f(C)\,dC \\ &= E\{\mu_{C_L}(C)\},\end{aligned} \tag{2}$$

where $\mathbb{R}^n$ is Euclidean n-space, $\mu_{C_L}(C)$ is the MF of the fuzzy event low conservativeness, C is a point in $\mathbb{R}^n$, P is a probability measure over $\mathbb{R}^n$, and f(C) is the pdf of the random variable (RV) C. We see that the probability of a fuzzy event is the expectation of its MF, as noted by Zadeh (1968: 47), and so (2) gives the expected value of low conservativeness, $E\{\mu_{C_L}(C)\}$.

In this instance, since we are working in the 1-D space of low conservativeness defined on $[0, C_S)$, the probability of low conservativeness is:

$$\tilde{P}(C_L) = \int_0^{C_S} \mu_{C_L}(C)\,f(C)\,dC \tag{3}$$

where $\mu_{C_L}(C)$ is given by (1).

2.1.2 Comment

The previous subsection merged fuzzy variables with random variables, and was a natural segue into a discussion of fuzzy random variables (FRVs), that is, random variables whose values are fuzzy numbers, and their implementation. However, except for some conceptual observations, such a discussion is beyond the scope of this article. The interested reader will find an overview of FRVs, from an actuarial perspective, in Shapiro (2009), and discussions of such topics as the expected value, variance, covariance and correlation of FRVs in Kwakernaak (1978, 1979), Kruse and Meyer (1987), Puri and Ralescu (1986), Körner (1997), Watanabe and Imaizumi (1999), Feng et al. (2001), Näther (2001), Couso and Dubois (2009) and Fullér et al (2010).

[2] Adapted from Zadeh (1968) and Suresh and Mujumdar (2004).

2.2 Sufficiency

In Figure 2[3] we look at sufficiency from two different perspectives: one is from the perspective of a retiree looking at the probability of sufficient funds, and the other is the same retiree looking at sufficiency from the point of view of being a member of the retiree group that has sufficient funds.

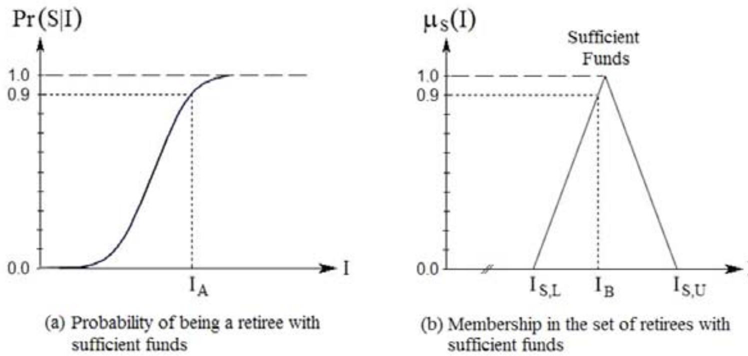

Figure 2: Two views of sufficient funds

In the figure S denotes sufficient funds. In Figure 2(a), I_A denotes the value of the classification index for the probability of sufficient funds. In Figure 2(b): the triangle, with support, whose lower and upper bounds are $I_{S,L}$ and $I_{S,U}$, respectively, represents the fuzzy set of sufficient funds; and $\mu_S(I)$, a value between zero and one, denotes the MF or GOM in the set of sufficient funds associated with the classification index I.

For comparison purposes, Figure 2(a) shows a 0.9 probability of sufficient funds, while Figure 2(b) shows a 0.9 GOM of the retiree in the group of retirees that have sufficient funds.

In Figure 3 we see that the notion of sufficient is one of degree. We could have, as in the right curve, sufficient funds, or, just to the left of that, just less than sufficient funds, and the individual can be a member in either, or both, of these groups. Thus, the individual might be considered to have a GOM of 0.9 in the set of retirees with sufficient funds, according to some criteria, and a GOM of 0.1 in the set of retirees that have just less than sufficient funds, according to other criteria.

[3] Adapted from Shapiro (2009, Figure 2)

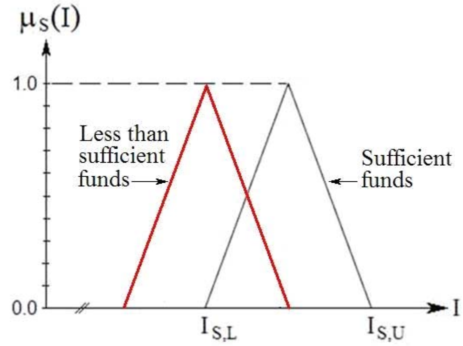

Figure 3: Sufficient funds v. less than sufficient funds

Figure 4 shows a representation of how a fuzzy variable and a random variable can be merged to form a FRV, or, more explicitly, in this case, a random fuzzy set.[4] The top of the figure is the MF for the set of retirees with sufficient funds and the bottom of figure is the probability that the RV funds, F, falls between F_L and F_U, which are comparable to $I_{S,L}$ and $I_{S,U}$, respectively. Similarly, we would represent the FRVs associated with the adjacent fuzzy sets.

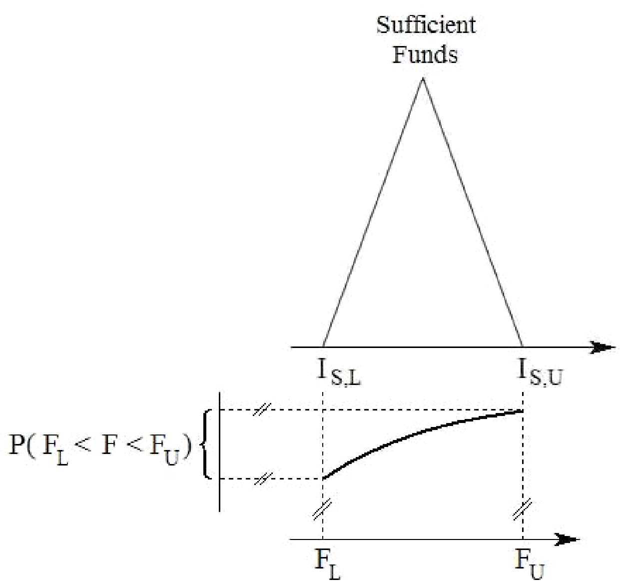

Figure 4: Sufficient funds as a random fuzzy set

The probability of sufficient funds proceeds in a manner similar to the development of (3), except that here we are working with a triangular fuzzy number (TFN). The essential difference is that the probability of each side of the TFN must be separately computed.

[4] The interpretation of a fuzzy random variable as a random fuzzy set was due to Puri and Ralescu (1986).

2.3 A Sense of Security

Figure 5[5] is concerned with the relationship between retirement funds and a sense of security for the retiree. To convey the idea, the funds, whose support ranges from $1M to $2M, are connected to the MF for the high sense of security for the individual. In this case, if the retirement funds are $1.5M, this individual would have a GOM of 75% in the group of retiree with a high sense of security.

Of course, the actual GOM depend on the characteristics of the retiree's actual MFs. Moreover, the probability that the retirement assets will not be compromised needs to be taken into account.

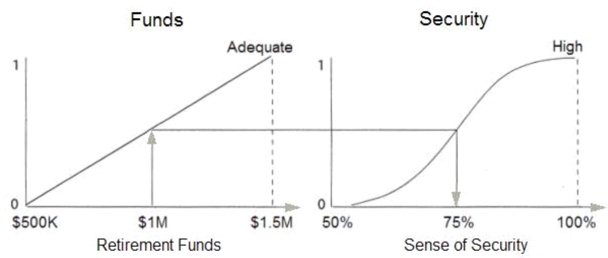

Figure 5: Retirement Funds versus a Sense of Security

In Figure 6, we extend this notion of adequate funds and this sense of security by depicting the MFs for the low, medium, and high sense of security, and attaching probabilities (p_L, p_M, p_H, respectively) to each of those MFs. In this way, we generate random fuzzy sets for each of these possibilities.

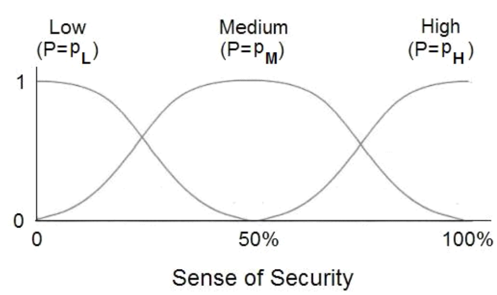

Figure 6: Sense of security as a random fuzzy set

[5] Adapted from Cox (1999: 88, Figure 3.17).

2.4 Preservation of Principal

One particularly relevant issue during a downturn in the market is that of preservation of principal. The concern, of course, is that retirees who are over-invested (a fuzzy term) in equities during such a time could have their retirement funds depleted to such an extent that they would not be able to recover. This subsection shows how preservation of principal can be represented as a MF, and how that MF can be implemented.

Figure 7 depicts one way to represent preservation of principal as a MF.

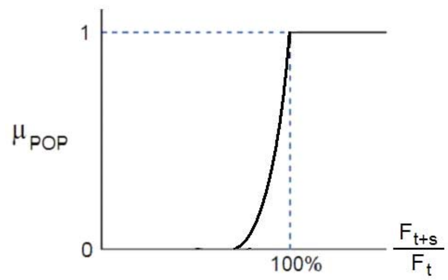

Figure 7: Preservation of principal

As indicated, the abscissa shows the relative principal, the ratio of the retirement fund at time t+s, s>0, F_{t+s}, to the fund at time t, F_t, and the ordinate shows the preservation of principal MF, μ_{POP}. If the relative principal is at 100%, or better, it is preserved, and the MF has a value of 1. However, there is a steep fall in the MF as the assets decrease below 100 percent of the principal. If it something less than 100%, then the principal is not preserved and the question is at what point do the assets fall out side the "preservation of principal" membership set.

The significance of the extent to which the principal is not preserved is a subjective determination of the retiree and should be a key consideration insofar as how the assets are allocated.

Figure 8[6] shows how the constraint of preservation of principal might be implemented using a goal based on expected return on assets. The analysis is from a prospective perspective.

[6] Adapted from Lemaire (1990, Figure 2).

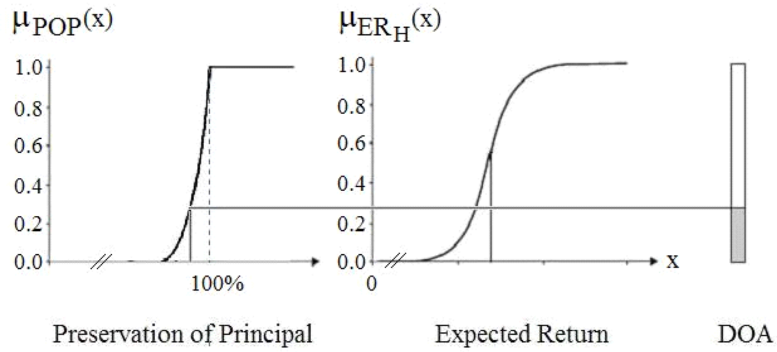

Figure 8: Preservation of principal v. high expected return

Again, there is the same sort of curve for the MF of the preservation of principal, and now there is a MF for the high expected return on assets, μ_{ER_H}. Moreover, there is a connection between the two because, as the expected return on assets gets larger, the percentage of the principal expected to be preserved decreases. Taking the intersection of the two MFs, the degree of applicability, DOA, falls to a lower level as the expected return goes up. This follows because the controlling factor in the interval considered is the preservation of principal, not the expected return.

3. Comment

The purpose of this article has been to present some preliminary observations regarding fuzzy post-retirement concepts, with a focus on fuzzy solvency issues from a retiree's perspective. The topics addressed included conservativeness, sufficiency, a sense of security, and preservation of principal. Of course, this list is far from exhaustive. Moreover, the analysis generally was in conceptual rather than technical terms, and was meant only to give the reader some insights into the issues involved. Nonetheless, to the extent that this article provides an impetus for further study into fuzzy post-retirement concepts, it will have served its purpose.

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