SOCIETY OF ACTUARIES®

Experience Study Rate Errors

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Caveat and Disclaimer

TABLE OF CONTENTS

Chapter 1: Introduction **4**

Chapter 2: Rate-Year Studies **5**
§2.1 Annual Rates and Forces 5
§2.2 Annual Rate Method 6
§2.3 Annual Force Method 6
§2.4 Fractional Rates and Forces 7
§2.5 Fractional Rate Method 8
§2.6 Fractional Force Method 9

Chapter 3: Calendar-Year Studies **11**
§3.1 Partial Age Notation 11
§3.2 Rate Distributions 12
§3.3 Annual Rate Method for Partial Ages 13
§3.4 Annual Force Method for Partial Ages 16

Chapter 4: Study Cohorts **19**
§4.1 Exposure by Cohort 19
§4.2 Exposure Weights 23
§4.3 Increasing Cohorts 24
§4.4 Increasing Cohort Model 25
§4.5 Cohort Periods 26
§4.6 Fractional Methods 29

Chapter 5: Actual Rate Distributions **31**
§5.1 Increase in Force 32
§5.2 Sample Table Relative Gradients 33
§5.3 Linear Force Distribution 35
§5.4 Bi-Linear Force Distribution 36
§5.5 Sample Linear Force Distributions 37

Chapter 6: Partial Age Errors **39**
§6.1 Generalized Error Formula 39
§6.2 Sample Age Distributions 45
§6.3 Sample Table Errors 46
§6.4 Projected Sample Age Errors 49
§6.5 Hybrid Annual Rate Method 51
§6.6 Increasing Cohorts 52
§6.7 Error Formula Derivation 54

Chapter 7: Annual Force with Weighted Exposure **57**

Chapter 8: Glossary **61**
§8.1 Terminology 61
§8.1 Notation 62

About the Society of Actuaries **65**

Chapter 1: Introduction

In 2016, I co-wrote a paper for the SOA that explored many calculations related to experience studies. That paper touched on errors that result from the application of various exposure methods in a calendar-year study. These errors arise from the use of two partial rate years, such as policy years or years of age, to cover one calendar year. This paper examines those errors much more closely and, in the end, suggests ways to minimize them.

To examine the absolute and relative errors arising from the different annual exposure methods, it is necessary to adapt them to handle partial rate years. This is accomplished by developing equivalent fractional year exposure methods using the mortality distributions implicit in the annual exposure methods. The results are compared using a continuous mortality distribution as the measurement standard.

Bringing together both annual and fractional-year approaches for the various exposure methods goes well beyond existing actuarial notation. **Chapter 8** is a glossary of the new terminology and notation used. It is recommended that you print the glossary and keep it handy as you read this paper.

This paper is organized as follows:

Chapters 2 and 3 review the main exposure methods covered in the original paper. **Chapter 2** introduces the methods for rate years, i.e., years that run from one birthday to the next or from one policy anniversary to the next. **Chapter 3** applies the exposure methods introduced in Chapter 2 to calendar-year studies. This introduces the use of partial ages or partial rate years to span a full year of age partitioned by the end of the calendar year. Partial rate years are the source of all exposure errors.

Chapter 4 looks at how the errors arising from the exposure methods accumulate in a study due to the layering of different cohorts in an experience study. A cohort is a group of lives a group of lives born in the same year. For select studies, the cohort is a group of lives born in the same year who purchased policies in the same year. In a multi-calendar-year study, multiple cohorts contribute exposure to the rates under study, such as rates by attained age or by issue age and policy year.

Chapter 5 introduces two mortality distributions that can be used to test exposure methods: The linear force and the bi-linear force distribution. Both distributions reproduce the annual mortality rate and assume that the force of mortality changes linearly over the year of age. The linear distribution does not enforce continuity from age to age, while the bi-linear distribution ensures continuity by using two linear segments within each year.

Chapter 6 uses the linear force distribution to estimate the partial year (or partial age) errors associated with three exposure methods. Using first-order approximations, a simple, generalized error formula is developed that applies to the three exposure methods. Distributions of fractional rates are calculated for the three exposures methods and compared to results from the bi-linear force distribution. A hybrid exposure method is introduced, which uses one exposure methods for the partial years at the beginning of a multi-calendar-year study and a different exposure method at the end of the study. The generalized error formula is used to examine the errors associated with the hybrid exposure method and for a single-year study and for a multi-year study with cohorts increasing in size.

The paper concludes with **Chapter 7**, which describes a weighted-exposure method that can be used to eliminate the errors associated with exposure for partial rate years or partial ages.

Chapter 2: Rate-Year Studies

The **rate year** is the period for which mortality rates are calculated in a rate-year mortality study. The rate year starts at one anniversary and ends at the next anniversary; both anniversaries must fall within the study period. The anniversary is typically the anniversary of birth, policy issue, or pension plan entry. Partial rate years running from the study start date to the next anniversary are excluded from a rate-year study. Similarly, partial rate years running from the last anniversary in the study to the study end date are excluded.

A mortality study calculates the probability of death, or mortality rate, for lives active at the start of each age. The traditional or actuarial exposure method calculates the mortality rate for an age as the number of deaths for the age divided by the exposure for the age. As mortality is a continuous decrement, an alternative approach is to use the average number of lives in force during the year of age to calculate the average force of mortality. The average force can then be converted into a mortality rate. Later in this section, we will show that these two approaches produce the same annual mortality rate.

For ease of reference, these two approaches are named and summarized below. Note that the average number of lives in force during the year of age is equivalent to the amount of time all lives are active during the year of age.

- The **Annual Rate Method** calculates the mortality rate for an age as the number of deaths for the age divided by exposure equal to the number of lives in force at the start of the age. This assumes that mortality is the only decrement.
- The **Annual Force Method** calculates the average force for an age as the number of deaths for the age divided by the amount of time all lives are active during the year of age. The mortality rate is then calculated from the annual force using §2.1 (1) i.e., formula (1) in section §2.1.

The mortality rate can also be calculated for fractional ages within an age. For example, a year of age could be examined by calculating twelve monthly mortality rates, one for each monthly period during the year of age. With a sufficiently large number of deaths, the fractional rates would allow the monthly pattern of mortality over the year of age to be analyzed.

- The **Fractional Rate Method** calculates the fractional mortality rate for each fractional age as the number of deaths that occurred during the fractional age divided by the number of lives in force at the start of the fractional age. The mortality rate for the full age is then calculated from the fractional mortality rates within the year of age using §2.4 (3).
- The **Fractional Force Method** calculates the fractional average force for each fractional age as the number of deaths that occurred during the fractional age divided by the average number of lives in force during the fractional age. The annual force for the full age is then calculated from the fractional average forces within the year of age using §2.4 (4), and the mortality rate is calculated using §2.1 (1).

In §2.1, the notation for annual rates and annual forces is introduced, as well as formulas that convert one to the other. Sections §2.2 and §2.3 show how annual rates and annual forces, respectively, are calculated. The final three sections, §2.4, §2.5 and §2.6, introduce notation and formulas for fractional rates and forces.

§2.1 Annual Rates and Forces

Traditional actuarial notation is used for the annual mortality rate and the annual average force of mortality:

q_x: The annual mortality rate from exact age x to $x+1$, and

$\bar{\mu}_x$: The annual average force of mortality from exact age x to $x+1$.

The annual average force notation includes a bar to distinguish it from the instantaneous force, μ_x at exact age x, or, more generally, μ_{x+t} at exact age $x+t$.

The relationship between the annual rate and average force is:

$$q_x = 1 - e^{-\bar{\mu}_x}, \tag{1}$$

and conversely

$$\bar{\mu}_x = -\log_e(1 - q_x). \tag{2}$$

§2.2 Annual Rate Method

To calculate the annual rate for a full year of age x, from exact age x to $x+1$ the exposure, deaths and annual rate are defined as:

E_x: The annual exposure for lives from exact age x to $x+1$.

- One year of exposure is assigned to lives active from exact age x to $x+1$, and
- One year of exposure is assigned to deaths between exact age x and $x+1$.
- A fraction of a year's exposure is assigned from the start of the year of age to the date of decrement, for decrements other than death, between exact age x and $x+1$. Note that decrements other than death are ignored here, but the full exposure definition is included for completeness.

d_x: The number of deaths between exact age x and $x+1$.

q_x: The annual mortality rate from exact age x to $x+1$.

The annual mortality rate is equal to number of deaths divided by annual exposure:

$$q_x = d_x / E_x. \tag{1}$$

§2.3 Annual Force Method

To calculate the annual force for a full year of age x, from exact age x to $x+1$, the exposure and annual force are defined as:

E_x^F: The annual exposure for Force (note the "F" superscript) for lives from exact age x to $x+1$.

- One year of exposure is assigned to lives active from exact age x to $x+1$, and
- A fraction of a year's exposure is assigned from the start of the year of age to the date of death, for deaths between exact age x and $x+1$.
- A fraction of a year's exposure is assigned from the start of the year of age to the date of decrement, for decrements other than death between exact age x and $x+1$. Note that decrements other than death are ignored here, but the full exposure definition is included for completeness.

$\bar{\mu}_x$: The average annual force from exact age x to $x+1$.

The annual average force is equal to the number of deaths divided by annual exposure for force:

$$\bar{\mu}_x = d_x/E_x^F \ . \qquad (1)$$

The annual mortality rate can then be calculated from the annual average force using §2.1 (1), i.e., formula (1) in subsection §2.1.

The annual force method is only appropriate for continuous, non-skewed distributions. To illustrate, consider a lapse study of 1000 lives with 10 lapses occurring at the end of the year and no other decrements. In this case, the rate and force exposures will be equal. That is,

$$E_x = E_x^F = 1000.$$

The annual rate method will calculate the correct lapse rate as:

$$q_x = d_x/E_x = 10/1000 = 0.1.$$

Because of the skewed distribution, the annual force will be incorrectly calculated as being equal to the annual rate:

$$\bar{\mu}_x = d_x/E_x^F = 10/1000 = 0.1.$$

When this annual force is converted to an annual rate using §2.1 (1), the result, using an F superscript to denote an annual rate calculated from an annual force, is not equal to the true annual rate.

$$q_x^F = 1 - e^{-0.1} = 0.0952 \neq q_x \ .$$

§2.4 Fractional Rates and Forces

We will use the following notation for fractional ages:

P: The number of periods per year. For monthly, $P = 12$.

f: The length of each fractional period, which equals $1/P$. For monthly, $f = 0.08333$.

s: The starting point of a fractional period. For monthly, the first three fractional periods correspond to $s = 0, f$, and $2f$, which in turn equal 0, 0.0833 and 0.1667. For the year of age x, the first three fractional periods run from ages $x + 0$ to $x + f$, from $x + f$ to $x + 2f$, and from $x + 2f$ to $x + 3f$.

The notation for the fractional mortality rate and average force incorporates an f subscript:

${}_fq_{x+s}$: The fractional mortality rate from exact age $x + s$ to $x + s + f$, and

${}_f\bar{\mu}_{x+s}$: The fractional average force of mortality from exact age $x + s$ to $x + s + f$.

The relationships between the fractional rate and fractional force are identical to Formulas §2.1 (1) and §2.1 (2), substituting fractional for annual rate and force:

$$ {}_fq_{x+s} = 1 - e^{-{}_f\bar{\mu}_{x+s}} \ , \qquad (1)$$

and conversely,

$$ {}_f\bar{\mu}_{x+s} = -\log_e\left(1 - {}_fq_{x+s}\right). \qquad (2)$$

Unlike annual studies, fractional studies provide direct insight into the mortality distribution over the year, while still returning the annual rate. The fractional mortality distribution is composed of the series of fractional rates or forces that for a full year of age. That is,

$_fq_x, {}_fq_{x+f}, {}_fq_{x+2f} \cdots, {}_fq_{x+(P-1)f}$, and

$_f\bar{\mu}_x, {}_f\bar{\mu}_{x+f}, {}_f\bar{\mu}_{x+2f} \cdots, {}_f\bar{\mu}_{x+(P-1)f}$.

The annual mortality rate is calculated as one minus the probability of surviving through all P periods. The probability of surviving each period is one minus the fractional mortality rate for the period:

$$q_x = 1 - \Pi_1^P\left(1 - {}_fq_{x+s}\right). \tag{3}$$

The annual force is the unweighted sum of the fractional forces:

$$\bar{\mu}_x = \Sigma_1^P {}_f\bar{\mu}_{x+s}. \tag{4}$$

The fractional rate and force can be annualized to compare them to the annual rates and forces. The A superscript indicates the annualized fractional rate and an annualized fractional force, as follows:

$${}_fq_{x+s}^A = 1 - \left(1 - {}_fq_{x+s}\right)^{1/f}. \tag{5}$$

$${}_f\bar{\mu}_{x+s}^A = {}_f\bar{\mu}_{x+s}/f. \tag{6}$$

Annualized fractional rates or forces would be equal to the annual rate or force only if mortality is constant over the year. So, generally:

$${}_fq_{x+s}^A \neq q_x. \tag{7}$$

$${}_f\bar{\mu}_{x+s}^A \neq \bar{\mu}_x. \tag{8}$$

§2.5 Fractional Rate Method

To calculate the fractional rate for a fractional year of age, from exact age $x + s$ to $x + s + f$, the exposure, deaths and fractional rate are defined as:

$_fE_{x+s}$: The fractional exposure for lives from exact age $x + s$ to $x + s + f$.

- One period of exposure is assigned to lives active from exact age $x + s$ to $x + s + f$, and
- One period of exposure is assigned to deaths between exact age $x + s$ and $x + s + f$.
- A fraction of a period's exposure is assigned from the start of the fractional year of age to the date of decrement for decrements other than death between exact age $x + s$ and $x + s + f$. Note that decrements other than death are ignored here, but the full exposure definition is included for completeness.

$_fd_{x+s}$: The number of deaths between exact age $x + s$ and $x + s + f$.

$_fq_{x+s}$: The fractional rate from exact age $x + s$ to $x + s + f$.

The fractional mortality rate is equal to the number of deaths divided by fractional exposure:

$${}_fq_{x+s} = {}_fd_{x+s}/{}_fE_{x+s}. \tag{1}$$

Annual exposure can be calculated from fractional exposure and fractional deaths, as follows:

$$E_x = \Sigma_1^P \left(f_f E_{x+s} + \left(1-(s+f)\right) {}_f d_{x+s} \right). \qquad (2)$$

The above formula can be simplified by assuming that deaths are uniformly distributed, i.e. ${}_f d_{x+s} = f d_x$, which results in the following approximate formula:

$$E_x \approx f \Sigma_1^P {}_f E_{x+s} + ½(1-f) d_x . \qquad (3)$$

Using the above formula, the annual rate can be approximated using fractional exposure, as follows:

$$q_x \approx \Sigma_1^P {}_f d_{x+s} / \left(f \Sigma_1^P {}_f E_{x+s} + ½(1-f) d_x \right). \qquad (4)$$

§2.6 Fractional Force Method

To calculate the fractional force for a fractional year of age x, from exact age $x+s$ to $x+s+f$, the exposure and fractional force are defined as:

${}_f E_{x+s}^F$: The fractional exposure for force for lives from exact age $x+s$ to $x+s+f$.

- One period of exposure is assigned to lives active from exact age $x+s$ to $x+s+f$, and
- A fraction of a period's exposure is assigned from the start of the year of age to the date of death for deaths between exact age $x+s$ and $x+s+f$.
- A fraction of a period's exposure is assigned from the start of the year of age to the date of decrement for decrements other than death between age $x+s$ and $x+s+f$. Note that decrements other than death are ignored here, but the full exposure definition is included for completeness.

${}_f \bar{\mu}_{x+s}$: The fractional force from exact age $x+s$ to $x+s+f$.

The fractional force is the number of deaths divided by fractional exposure for force:

$${}_f \bar{\mu}_{x+s} = {}_f d_{x+s} / {}_f E_{x+s}^F . \qquad (1)$$

The fractional mortality rate can then be calculated from fractional force using §2.4 (1):

$${}_f q_{x+s} = 1 - e^{-{}_f \bar{\mu}_{x+s}} . \qquad (2)$$

Annual exposure can be calculated using the sum of the fractional exposures. It is necessary to multiply the sum by f to adjust from fractional exposure (i.e., exposure of 1 per fractional period) to annual exposure (i.e., exposure of 1 per year):

$$E_x^F = f \Sigma_1^P {}_f E_{x+s}^F . \qquad (3)$$

The results of a fractional force study can be used to directly calculate an annual force, as follows:

$$\bar{\mu}_x = d_x / E_x^F = \Sigma_1^P {}_f d_{x+s} / \left(f \Sigma_1^P {}_f E_{x+s}^F \right) . \qquad (4)$$

Note that this relationship only holds true for continuous, non-skewed distributions, as was demonstrated in §2.3.

We will denote fractional exposure weights for each of the P fractional periods (i.e., for $s = 0, f, 2f, .., (P-1)/P$) as:

${}_f \alpha_{x+s}^F$: The fractional exposure weight from exact age $x+s$ to $x+s+f$.

The fractional exposure weight will be defined as the ratio of fractional exposure for the period to the sum of the fractional exposures over the year:

$$ {}_f\alpha_{x+s}^F = {}_fE_{x+s}^F / \Sigma_1^P {}_fE_{x+s}^F \; . \tag{5} $$

Substituting (3), the fractional exposure weight can now be expressed as annualized fractional exposure divided by annual exposure:

$$ {}_f\alpha_{x+s}^F = f \, {}_fE_{x+s}^F / E_x^F \; . \tag{6} $$

Using (1) to substitute for ${}_fd_{x+s} = {}_fE_{x+s}^F \, {}_f\bar{\mu}_{x+s}$ into (4), and then simplifying using (6), results in:

$$ \bar{\mu}_x = \Sigma_1^P \, {}_fE_{x+s}^F \, {}_f\bar{\mu}_{x+s} / \left(f \Sigma_1^P \, {}_fE_{x+s}^F\right) = \Sigma_1^P \, {}_f\alpha_{x+s}^F \left({}_f\bar{\mu}_{x+s}/f\right) . \tag{7} $$

As the annual force is also equal to the unweighted sum of the fractional forces, per §2.4 (4), this gives the following identity:

$$ \Sigma_1^P \, {}_f\alpha_{x+s}^F \left({}_f\bar{\mu}_{x+s}/f\right) = \Sigma_1^P \, {}_f\bar{\mu}_{x+s} \; . \tag{8} $$

That is, the sum of the exposure-weighted annualized fractional forces is equal to the sum of the fractional forces over the year. This relationship only holds for continuous non-skewed distributions and could be used to define a non-skewed distribution.

Chapter 3: Calendar-Year Studies

Unless the rate year is a calendar year, each year of age will include time from two consecutive calendar years. Looked at another way, a multi-calendar-year study period will include partial ages at the start and end of the study. That is, the lives active at the start (or end) of the study will enter (or leave) the study part way through a year of age. A multi-calendar-year study can be reconstructed as a series of one-year calendar-year studies where deaths and exposure for each calendar year are calculated separately. This would enable the study of calendar-year trends within the study period.

The exposure for a partial age assumes that the risk of death is proportional to the length of the partial age. For lives active throughout the partial age, this is simply the length of time in years of the partial age. For the exposure on deaths in the period, there is no impact on the annual force method, as exposure is assigned to the date of death, but for the annual rate method, there are two approaches to calculating exposure.

- **Traditional exposure**: Exposure on death up to the end of the year of age is allocated to the partial age in which death occurred.
- **Distributed exposure**: Exposure on death is <u>distributed</u> between the two partial ages that constitute the rate year, as follows:
    - Exposure from the date of death to the end of the partial age in which death occurred is assigned to the partial age in which death occurred.
    - If death occurred in the first partial age, then exposure from the end of the first partial age to the end of the rate year is assigned to the second partial age, which completes the rate year.

The last bullet point means that some exposure from each death occurring during a first partial age ending at the study's start date will be included in the study. When first implementing the distributed exposure method, deaths prior to the study start date may not be available. This could result in a **hybrid exposure method** using traditional exposure for partial ages at the start of the study and distributed exposure for partial ages at the end of the study.

All the exposure methods assume that risk is proportional to the time exposed. This assumption of proportional risk results in imputed distributions of mortality rates for the rate year. For each exposure method, these imputed distributions are the source of errors when partial age data is used to calculate annual rates.

§3.1 Partial Age Notation

A full year of age, from exact age x to $x+1$, is divided into two partial ages, one ending and one beginning at the calendar year end. Let t be the time from the start of an age, i.e. the birthday or policy anniversary, to the end of the calendar year. Then the first partial age is from exact age x to $x+t$, and the second partial age is from exact age $x+t$ to $x+1$.

The previous notation for fractional ages can be adapted to partial ages: A partial age will begin at time s in the year of age x and run for a period f, i.e. from exact age $x+s$ and $x+s+f$, where $s<1$ and $f \leq 1-s$. Hence a partial age refers to any period within an age, while a fractional age refers to an age being divided into equal periods. For two partial ages arising from a full year of age being partitioned by calendar year, then $P=2$ and the two partial ages are defined as:

- First partial age from exact age x to $x+t$, equivalent to $s=0$ and $f=t$, and
- Second partial age from exact age $x+t$ to $x+1$, equivalent to $s=t$ and $f=1-t$.

The same partial age notation can be applied to both the first and second partial ages, thereby removing the need for duplicate formulas. In the following, either notation will be used as appropriate.

§3.2 Rate Distributions

For the annual rate method with traditional exposure, the partial rate from exact age $x+t$ to $x+1$ is assumed to be proportional to the annual rate. That is,

$$ {}_{1-t}q_{x+t} = (1-t)q_x \,. \tag{1} $$

This is called the Balducci hypothesis and results in mortality decreasing over the year of age. This runs counter to actual mortality trends at most ages.

For the annual rate method with the distributed exposure, the partial rate from exact age x to $x+t$ is assumed proportional to the annual rate. That is,

$$ {}_{t}q_x = tq_x \,. \tag{2} $$

This is equivalent to a uniform distribution of deaths assumption and results in mortality increasing over the year of age.

For the hybrid method, the partial age at the start of the study assumes that mortality is decreasing in the year of age, while the partial age at the end of the study assumes that mortality is increasing in the year of age.

For the annual force method, the partial rate from exact age x to $x+t$ is assumed equal to the annual rate expressed as a partial rate. That is,

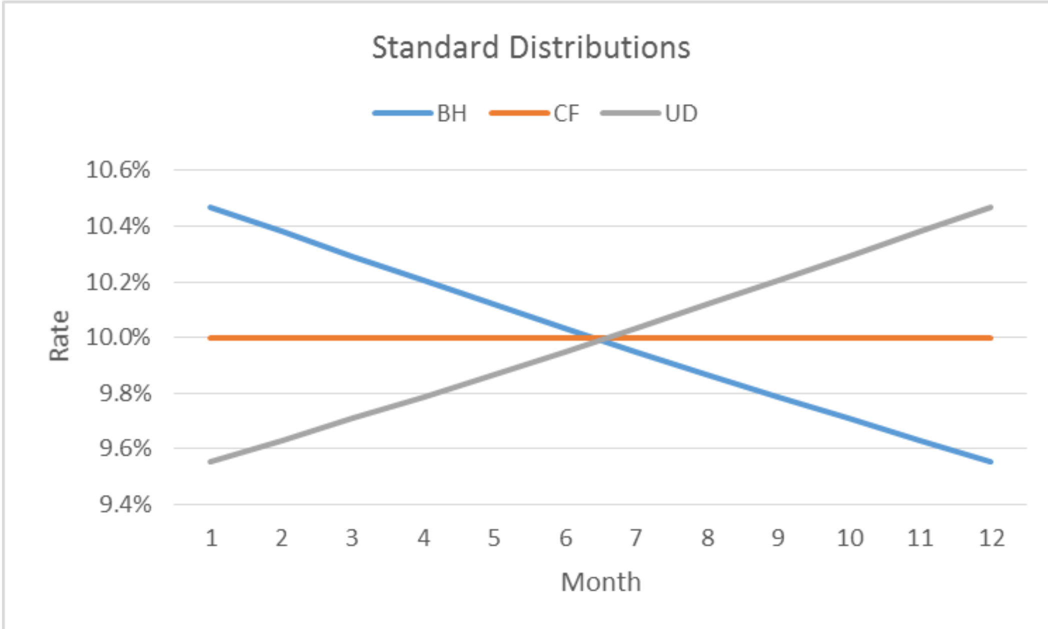
$$ {}_{t}q_x = 1-(1-q_x)^t \,. \tag{3} $$

Expressed in terms of force, the partial average force from exact age x to $x+t$ is assumed proportional to the annual average force. That is,

$$ {}_{t}\bar{\mu}_x = t\bar{\mu}_x \,. \tag{4} $$

This is equivalent to mortality being constant over the year of age. This will be referred to as the constant force assumption.

The graph below shows the annualized monthly rates for a 10% rate using the Balducci hypothesis (BH), uniform distribution of deaths (UD), and constant force (CF) assumptions.



As mortality increases with age, and hence increases within a year of age, the traditional exposure method is considered flawed as it assumes that mortality is decreasing over the year of age, while the distributed exposure method can be considered an improvement, as mortality is assumed increasing for both starting and ending partial ages. For the hybrid method, mortality is increasing only for the ending partial age. Similarly, the constant force method may be considered somewhat preferable as the constant mortality assumption is "closer" to increasing mortality than the decreasing mortality of the traditional method.

§3.3 Annual Rate Method for Partial Ages

The annual rate method notation for partial ages is given below. To distinguish notation for partial ages from that for fractional ages, a subscript for the period f will be added to the right-hand side of the symbol, instead of to the left-hand side. As exposure is calculated in years, it will be referred to as annual exposure despite being calculated for a partial age. The resulting mortality rate is also considered an annual rate. References to partial exposure and partial rates will refer to exposure and rates that are calculated for a partial year of age. In both cases, rates are calculated as deaths divided by exposure.

To calculate the annual rate for a partial year of age, from exact age $x+s$ to $x+s+f$, the following variables will be used. Annual exposure for each exposure method is defined below, after the annual mortality rate.

$E_{x+s,f}$: The annual exposure for partial age $x+s$ to $x+s+f$.

$d_{x+s,f}$: The number of deaths for partial age $x+s$ and $x+s+f$.

$q_{x+s,f}$: The annual mortality rate for partial age $x+s$ to $x+s+f$.

The annual mortality rate for a partial age is its deaths divided by its annual exposure:

$$q_{x+s,f} = d_{x+s,f}/E_{x+s,f}\,. \tag{1}$$

Two different methods are used to calculate exposure for partial ages. The exposure and rate for each exposure method will be identified by its underlying mortality assumption: For traditional exposure, BH (Balducci hypothesis) will be used. For distributed exposure, UD (uniform distribution of deaths) will be used.

Traditional exposure is denoted and calculated as:

$E^{BH}_{x+s,f}$: The annual exposure for lives from exact age $x+s$ to $x+s+f$.

- f years of exposure is assigned to lives active from exact age $x+s$ to $x+s+f$, and
- $1-s$ years of exposure, i.e., exposure to the end of the year of age, is assigned to deaths between exact age $x+s$ and $x+s+f$.

Comparing the above definition to that for fractional exposure, given in §2.5, the annual exposure for the partial age can be expressed in terms of the partial exposure, as follows.

$$E^{BH}_{x+s,f} = f\,{}_fE_{x+s} + \big(1-(s+f)\big)\,{}_fd_{x+s}\,. \tag{2}$$

Substituting (2) into (1) gives:

$$q^{BH}_{x+s,f} = \left({}_fq_{x+s}/f\right)/\Big(1+\ \big(1-(s+f)\big)\left({}_fq_{x+s}/f\right)\Big). \tag{3}$$

Using the Taylor expansion, i.e., $1/(1+X) = 1 - X + X^2 - X^3 + \cdots$, the annual rate using Traditional exposure can be estimated by ignoring third and higher powers as:

$$q^{BH}_{x+s,f} \approx {}_fq_{x+s}/f + \big(\mathrm{s}-(1-f)\big)({}_\mathrm{f}q_{x+s}/f)^2\,. \tag{4}$$

Distributed exposure is denoted and calculated as:

$E^{UD}_{x+s,f}$: The annual exposure for lives from exact age $x+s$ to $x+s+f$.

- f years of exposure is assigned to lives active from exact age $x+s$ to $x+s+f$, and
- f years of exposure is assigned to deaths between exact age x and $x+s+f$. Note that this includes exposure for deaths between exact age x and $x+s$, which is the exposure that is "distributed."

The annual exposure for the partial age can be expressed in terms of the partial exposure, as:

$$E^{UD}_{x+s,f} = f\,{}_fE_{x+s} + f\,{}_sd_x\,. \tag{5}$$

Substituting (5) into (1) gives:

$$q^{UD}_{x+s,f} = \left({}_fq_{x+s}/f\right)/\Big(1+\ {}_fq_{x+s}\left({}_sd_x/{}_fd_{x+s}\right)\Big). \tag{6}$$

In (6), the ratio of ${}_sd_x$ to ${}_fd_{x+s}$ can be replaced by s/f, assuming that deaths are approximately uniformly distributed. The annual rate can then be expressed in terms of the partial rate as:

$$q^{UD}_{x+s,f} \approx \left({}_fq_{x+s}/f\right)/\Big(1+\ s\left({}_fq_{x+s}/f\right)\Big)\,. \tag{7}$$

Once again using the Taylor expansion, the annual rate using Distributed exposure can be estimated by ignoring third and higher powers as:

$$q^{UD}_{x+s,f} \approx \left({}_fq_{\mathrm{x+s}}/f\right) - \ s\left({}_fq_{\mathrm{x+s}}/f\right)^2\,. \tag{8}$$

In the remainder of this subsection, we will examine the difference between a) the sum of annual exposures over the partial ages that constitute a year of age and b) annual exposure for the full year of age. That difference will allow us to analyze partial age rate errors. The formulas and relationships presented in the remainder of this subsection will apply equally to Traditional and Distributed exposure. For convenience, we will use the following notation:

- $E_{x+s,f}$ will be used to denote both Traditional ($E_{x+s,f}^{BH}$) and Distributed ($E_{x+s,f}^{UD}$) exposure.
- $q_{x+s,f}$ will be used to denote both Traditional ($q_{x+s,f}^{BH}$) and Distributed ($q_{x+s,f}^{BH}$) rates.

For a year of age split into P periods that are not necessarily equal, the annual exposure for the full year is the sum of the annual exposures for the partial ages that constitute the year of age. That is,

$$E_x = \Sigma_1^P E_{x+s,f} \ . \tag{9}$$

The annual rate for the full year of age is equal to the ratio of total deaths to total exposure. This, in turn, is equal to the ratio of the sum of partial age deaths to the sum of partial age exposures for the year.

$$q_x = d_x / E_x = \Sigma_1^P d_{x+s,f} / \Sigma_1^P E_{x+s,f} \, . \tag{10}$$

The partial age exposure weight will be denoted as:

$\alpha_{x+s,f}$: The exposure weight for partial age $x+s$ to $x+s+f$.

The partial age exposure weight is defined as the ratio of annual exposure for the partial age to the annual exposure for the year of age in which the partial age falls:

$$\alpha_{x+s,f} = E_{x+s,f} / E_x \tag{11}$$

Substituting (11) into (10) gives the annual rate over the full year of age as sum of the exposure-weighted annual rates for the partial ages.

$$q_x = \Sigma_1^P \alpha_{x+s,f} q_{x+s,f} \tag{12}$$

The annual rate for a partial age is only an estimate of the annual rate for the full year of age. That is,

$q_{x+s,f} \neq q_x$, if $s > 0$ or $f < 1$.

However, if the distribution of deaths within each partial age were to match the assumed distribution of deaths underlying the exposure method, then the annual rate for each partial age would equal the annual rate for the full year of age.

The difference between the annual rate for a partial age and the annual rate for the full year of age is the error in the annual rate for that partial age. We will denote that error as follows:

$\varepsilon_{x+s,f}$: The error in the annual rate for partial age $x+s$ to $x+s+f$.

The error is the difference between the annual rate for the partial age and the annual rate for the full year of age:

$$\varepsilon_{x+s,f} = q_{x+s,f} - q_x \ . \tag{13}$$

Using (13) to substitute $q_x = q_{x+s,f} + \varepsilon_{x+s,f}$ into (12) gives:

$$q_x = \Sigma_1^P \alpha_{x+s,f} \left(q_x + \varepsilon_{x+s,f} \right) = q_x + \Sigma_1^P \alpha_{x+s,f} \varepsilon_{x+s,f} \ . \tag{14}$$

Hence the exposure-weighted sum of the partial age errors over the full year is equal to zero:

$$\Sigma_1^P \alpha_{x+s,f} \varepsilon_{x+s,f} = 0\,. \tag{15}$$

So, for a full year of age divided between two calendar years, the exposure-weighted errors in each calendar year are equal in absolute value and opposite in sign. That is,

$$\alpha_{x,t-1} \varepsilon_{x,t} = -\alpha_{x+t,t} \varepsilon_{x+t,1-t}\,. \tag{16}$$

§3.4 Annual Force Method for Partial Ages

The exposure and rate notation and terminology introduced in §3.3, Annual Rate Method for Partial Ages, is continued in this subsection. The Annual Force Method for Partial Ages assumes that the force is constant over the year of age. The superscript CF, for "Constant Force," has not been added to the notation, as no other force assumption is used in this subsection. In later sections, superscript CF will be added as required.

The exposure for annual force for a partial age is denoted and calculated as:

$E_{x+s,f}^F$: The annual exposure for **F**orce (denoted by the "F" superscript) for partial age $x+s$ to $x+s+f$.

- f years of exposure is assigned to lives active from exact age $x+s$ to $x+s+f$, and
- A fraction of a year's exposure is assigned from the start of the partial age to the date of death for deaths between exact age $x+s$ and $x+s+f$.

The annual force for a partial age is denoted as:

$\bar{\mu}_{x+s,f}$: The annual force for partial age $x+s$ to $x+s+f$.

The annual force for a partial age is equal to deaths during the partial age divided by its exposure:

$$\bar{\mu}_{x+s,f} = d_{x+s,f} / E_{x+s,f}^F \tag{1}$$

The mortality rate for the partial age, calculated using the force of mortality, is:

$$q_{x+s,f} = 1 - e^{-\bar{\mu}_{x+s,f}} \tag{2}$$

Annual exposure for a partial age is equal to the fraction f times fractional exposure, as defined in §2.6:

$$E_{x+s,f}^F = f\, {}_fE_{x+s}^F \tag{3}$$

Comparing the definitions of exposure for a partial age's fractional force (see lead-in to §2.6 (1)) and the exposure defined above for a partial age's annual force (see lead-in to §3.5 (1)), it becomes clear that the partial age's fractional force will equal the partial age's annual force times f. This means a partial age's annual force is always equal to its annualized partial force, as shown below:

$$\bar{\mu}_{x+s,f} = {}_f\bar{\mu}_{x+s} / f = {}_f\bar{\mu}_{x+s}^A \tag{4}$$

For a year of age split into P periods that are not necessarily equal, the annual exposure for the full year of age is equal to the sum of the annual exposures for its constituent partial ages. It is also equal to the sum of its partial exposures weighted for time (f):

$$E_x^F = \Sigma_1^P E_{x+s,f}^F = \Sigma_1^P f\, {}_fE_{x+s}^F \tag{5}$$

Similarly, the annual force for the full year of age is equal to the sum of the annual forces for its partial ages weighted for time (f). It is also equal to the sum of the partial forces from its constituent partial ages:

$$\bar{\mu}_x = \Sigma_1^P f \bar{\mu}_{x+s,f} = \Sigma_1^P {}_f\bar{\mu}_{x+s} \tag{6}$$

The annual force of the full year of age is also equal to the sum of deaths over all of its partial ages divided by the sum of annual force exposure over all partial ages.

$$\bar{\mu}_x = d_x / E_x^F = \Sigma_1^P d_{x+s,f} / \Sigma_1^P E_{x+s,f}^F \tag{7}$$

The partial age force-exposure weight will be denoted as:

$\alpha_{x+s,f}^F$: The force-exposure weight for partial age $x + s$ to $x + s + f$

The partial age force-exposure weight is equal to the annual force exposure for the partial age divided by the annual force exposure for the year of age in which the partial age falls:

$$\alpha_{x+s,f}^F = E_{x+s,f}^F / E_x^F \tag{8}$$

Substituting (8) into (7) gives the annual force for the full year of age as the force-exposure-weighted sum of the annual forces for its constituent partial ages.

$$\bar{\mu}_x = \Sigma_1^P \alpha_{x+s,f}^F \bar{\mu}_{x+s,f} \tag{9}$$

From (6), the annual force is also equal to the time-weighted sum of the annual forces for the partial ages in the year, which leads to the following identity:

$$\Sigma_1^P \alpha_{x+s,f}^F \bar{\mu}_{x+s,f} = \Sigma_1^P f \bar{\mu}_{x+s,f} \tag{10}$$

That is, the sum of the force-exposure-weighted annual forces for the partial ages in a year is equal to the time-weighted sum of the annual forces for the partial ages in the year. This relationship only holds for continuous non-skewed distributions and could be used to define a non-skewed distribution.

The annual force for a partial age is usually not equal to the annual force for the full year of age:

$\bar{\mu}_{x+s,f} \neq \bar{\mu}_x$, if $s > 0$ or $f < 1$.

In the remainder of this subsection, we will examine the difference between a) the sum of annual forces over the partial ages that constitute a year of age and b) the annual force for the full year of age. That difference will allow us to analyze partial age annual-force errors.

The difference between the annual force for a partial age and the annual force for the full year of age is the error in the annual force for that partial age. We will denote that error using the Greek letter xi:

$\xi_{x+s,f}$: The error in the annual force for partial age $x + s$ to $x + s + f$.

The error is the difference between the annual force for the partial age and the annual force for the full year of age:

$$\xi_{x+s,f} = \bar{\mu}_{x+s,f} - \bar{\mu}_x \tag{11}$$

Using (11) to substitute $\bar{\mu}_x = \bar{\mu}_{x+s,f} + \xi_{x+s,f}$ into (9) gives us:

$$\bar{\mu}_x = \Sigma_1^P \alpha_{x+s,f}^F \left(\bar{\mu}_x + \varepsilon_{x+s,f}^F\right) = \bar{\mu}_x + \Sigma_1^P \alpha_{x+s,f}^F \xi_{x+s,f} \qquad (12)$$

From (12), we can deduce that the exposure-weighted sum of the partial age errors over the full year of age is equal to zero:

$$\Sigma_1^P \alpha_{x+s,f}^F \xi_{x+s,f} = 0 \qquad (13)$$

So, for a full year of age divided between two calendar years, the exposure-weighted errors in each calendar year are equal in absolute value and opposite in sign. That is,

$$\alpha_{x,t}^F \xi_{x,t} = -\alpha_{x+t,1-t}^F \xi_{x+t,1-t} \qquad (14)$$

The annual force method for partial ages can be more directly compared with the annual rate method by expressing the annual rate for a partial age in terms of the partial rate, as follows:

$$q_{x+s,f} = 1 - \left(1 - {}_f q_{x+\mathrm{s}}\right)^{1/f} \qquad (15)$$

Using the Taylor expansion, $(1+X)^a = 1 + aX + a(1-a)X^2/2 + \cdots$, and ignoring third and higher powers, we have:

$$q_{x+s,f} \approx {}_f q_{x+\mathrm{s}}/f - ½(1-f)\left({}_f q_{x+t}/f\right)^2 \qquad (16)$$

Conversely, as the annual rate for the partial age will be known, the partial rate can be calculated from the annual rate as:

$$_f q_{x+\mathrm{s}} = 1 - \left(1 - q_{x+s,f}\right)^f \qquad (17)$$

This can be expanded to show the relationship with respect to the first and second order terms:

$$_f q_{x+\mathrm{s}} \approx f q_{x+s,f} - ½ f(1-f)\left(q_{x+s,f}\right)^2 \approx f q_{x+s,f} \qquad (18)$$

Chapter 4: Study Cohorts

For purposes of this and later sections, a **study cohort** will be a group of lives born in the same year. Each cohort will contribute exposure and deaths to one or more attained ages in the study. In a study composed of N calendar years, a single cohort may contribute data for up to $N+1$ consecutive ages, i.e., to ages $x, x+1, .. x+N$. Looked at another way, multiple cohorts may contribute data to the same age. Therefore, multiple cohorts may contribute to the errors for partial ages. In this section, we will examine how cohort errors can partially offset one another. Defining the study cohorts with respect to the calendar year of birth is only appropriate if the study is using life-years to define the rates. For other rate-years, the cohort year would need to be adjusted to reflect the rate year. We will assume that lives enter the study at the start of a rate year or year of age except at the start of the study where they enter at the study start date.

For example, in a study of three calendar years, from 2012 to 2014, lives born in 1941 will contribute exposure and deaths to ages 70, 71, 72 and 73, as follows:

- Because lives born in 1941 turn 70 in 2011, age 70 will be a partial age that runs from the start of the study (January 1, 2012) to their 71st birthdays in 2012.
- The entirety of ages 71 and 72 will fall within the study period.
- Age 73 will be a partial age that runs from their birthdays in 2014 to the end of the study (December 31, 2014).

The following subsection, §4.1, will show cohorts spread across tables organized by study year and age to illustrate how exposures, rates and errors accumulate in a study. We will then explore simplifications for situations where cohorts are equal in size. The partial exposure weights by cohort year and across the study period are examined in §4.2. As studies with equal cohorts are rarely seen in practice, §4.3 derives a formula to investigate the impact of changing cohort sizes on the error flowing through to a study. §4.4 develops a simple model for cohorts that increase in size by the same annual amount. This model will be used in §6.6 to look at the magnitude of errors flowing into a study for a range of cohort increases.

§4.1 Exposure by Cohort

In the following table, columns for 2011 through 2015 and totals have rows that show the exposure contributed by ages 70 to 73. The years 2011 and 2015 are shaded because they are not part of the study: They have been included to show the exposure excluded from the study for ages 70 and 73. The totals for ages 71 and 72 contain a full year of exposure, as indicated by the second subscript, "1." The totals for ages 70 and 73 contain only a partial age of exposure, as indicated by the second subscript of "$1-t$" for age 70 and "t" for age 73. The time t will only be the same for lives with the same birthday. As birthdays will be distributed over the year, the time t will represent the average time to calendar year-end for the full cohort. For example, if birthdays are uniformly distributed over the year, $t = ½$. It will be assumed that the distribution of birthdays over the year will be the same for each cohort.

Exposure by Study Year and Age						
Age	2011	2012	2013	2014	2015	Total
70	$E_{70,t}$	$E_{70+t,1-t}$				$E_{70+t,1-t}$
71		$E_{71,t}$	$E_{71+t,1-t}$			$E_{71,1}$
72			$E_{72,t}$	$E_{72+t,1-t}$		$E_{72,1}$
73				$E_{73,t}$	$E_{73+t,1-t}$	$E_{73,t}$

Repeating the above format, the next table shows mortality rates from the same study. For ages 71 and 72, the rates in the Total column are the exposure-weighted sum of the rates from the contributing study years.

Mortality Rates by Study Year and Age						
Age	2011	2012	2013	2014	2015	Total
70	$q_{70,t}$	$q_{70+t,1-t}$				$q_{70+t,1-t}$
71		$q_{71,t}$	$q_{71+t,1-t}$			$q_{71,1}$
72			$q_{72,t}$	$q_{72+t,1-t}$		$q_{72,1}$
73				$q_{73,t}$	$q_{73+t,1-t}$	$q_{73,t}$

The following table shows the errors in the study's mortality rates. For ages 71 and 72, the errors in the total column are the exposure-weighted sum of the errors from the contributing study years.

Rate Errors by Study Year and Age						
Age	2011	2012	2013	2014	2015	Total
70	$\varepsilon_{70,t}$	$\varepsilon_{70+t,1-t}$				$\varepsilon_{70+t,1-t}$
71		$\varepsilon_{71,t}$	$\varepsilon_{71+t,1-t}$			0
72			$\varepsilon_{72,t}$	$\varepsilon_{72+t,1-t}$		0
73				$\varepsilon_{73,t}$	$\varepsilon_{73+t,1-t}$	$\varepsilon_{73,t}$

The above examples were based on a study with all lives born in 1941, so that arrays of exposure, rates and errors could be clearly shown by study year and age.

In practice, many years of birth are included in a mortality study, so we will expand the example to include four cohorts: birth years 1941 through 1944. Going forward, each age in the study will have exposure and death contributions from lives born in one to four consecutive years.

Focusing on age 70, four cohorts contribute to the age 70 rate in three different ways:

- Lives born in 1941 contribute a partial age 70, from the start of the study (January 1, 2012) to the 71st birthday in 2012.
- Lives born in 1942 and 1943 each contribute a full age 70, as each age 70 falls fully within the study.
- Lives born in 1944 turn 70 in 2014 and contribute only a partial age from their birthdays in 2014 to the end of the study period (December 31, 2014).

In summary, the mortality rate for age 70 is now based on data from two partial ages, for lives born in 1941 and 1944, and two full years of age, for lives born in 1942 and 1943.

In the remainder of this section, we will introduce and use variables that include the cohort year to which they apply. These variables will use the following three subscripts:

- $x + s$ is the starting exact age for the variable.
- f is the time interval or length of time to which the variable applies. For a full year of age, $f = 1$.
- CY is the cohort year to which the variable applies. For brevity, cohort years are numbered consecutively starting with 1 for each age. In the following examples for age 70, "1" indicates birth year 1941, "2" indicates birth year 1942, etc.

The following variables will be used to track the contributions from each cohort year:

$E_{x+s,f,CY}$: The exposure for lives from exact age $x + s$ to $x + s + f$ for cohort year CY.
$d_{x+s,f,CY}$: The number of deaths between exact age $x + s$ to $x + s + f$ for cohort year CY.
$q_{x+s,f,CY}$: The mortality rate from exact age $x + s$ to $x + s + f$ for cohort year CY.

The age 70 exposure contributions by cohort year for lives born in 1941 to 1944 are illustrated in the table below.

Exposure by Cohort Year					
Age	1941	1942	1943	1944	Total
70	$E_{70+t,1-t,1}$	$E_{70,1,2}$	$E_{70,1,3}$	$E_{70,t,4}$	E_{70}

The total exposure for age 70, E_{70}, is the sum of the exposures from each cohort. More generally, in a study composed of N calendar years, exposure for age x will have contributions from M cohorts, where M is no less than zero and no more than $N + 1$ or the total number of cohorts. For this example, age 70 has contributions from the maximum number of cohorts such that $M = N + 1 = 4$.

$$E_x = \Sigma_{CY=1}^{M} E_{x+s,f,CY} \qquad (1)$$

The age 70 mortality rates calculated for each cohort and for the study in total are shown in the next table:

Mortality Rates by Cohort Year					
Age	1941	1942	1943	1944	Total
70	$q_{70+t,1-t,1}$	$q_{70,1,2}$	$q_{70,1,3}$	$q_{70,t,4}$	q_{70}

The mortality rate across all cohorts, q_x, is the exposure-weighted sum of the mortality rates for each cohort:

$$q_x = \Sigma_{CY=1}^{M} \left(E_{x+s,f,CY}/E_x\right) q_{x+s,f,CY} \qquad (2)$$

The error in the mortality rates calculated for each cohort and for the study in total is:

Rate Errors by Cohort Year					
Age	1941	1942	1943	1944	Total
70	$\varepsilon_{70+t,1-t,1}$	0	0	$\varepsilon_{70,t,4}$	ε_{70}

The total error across all cohorts, ε_x, is the exposure-weighted sum of the errors for each cohort:

$$\varepsilon_x = \Sigma_{CY=1}^{M} \left(E_{x+s,f,CY}/E_x\right) \varepsilon_{x+s,f,CY} \qquad (3)$$

As the errors for the full years of age, such as 1942 and 1943 in the above example, are zero, the total error is the exposure-weighted sum of the errors from the first and last cohorts:

$$\varepsilon_x = \left(E_{x+t,1-t,1}/E_x\right)\varepsilon_{x+t,1-t,1} + \left(E_{x,t,N+1}/E_x\right)\varepsilon_{x,t,N+1} \tag{4}$$

We will extend §3.3 (11), i.e., formula (11) in Subsection §3.3, to identify cohorts for the full year of age x split between two calendar years: this results in two partial ages: (x, t) and $(x + t, 1 - t)$. As new entrants are assumed to enter at the start of the rate year, the exposure distribution within the year of age is dependent only on the distribution of deaths within the year. The partial age exposure weights for cohort CY are defined as partial age exposure divided by the full exposure for the age:

$$\alpha_{x,t,CY} = E_{x,t,CY}/E_{x,1,CY} \tag{5}$$

$$\alpha_{x+t,1-t,CY} = E_{x+t,1-t,CY}/E_{x,1,CY} \tag{6}$$

Next, we will assume that the distribution of deaths and exposure are the same for all cohorts. This may be a reasonable assumption for most cohorts in a mortality study by attained age.

Assuming the exposure distribution is the same across all cohorts, then the partial age exposure weights will be the same across all cohorts:

$$\alpha_{x,t,CY} = \alpha_{x,t} \tag{7}$$

$$\alpha_{x+t,1-t,CY} = \alpha_{x+t,1-t} \tag{8}$$

Substituting (7) and (8) into (5) and (6) and rearranging gives:

$$E_{x,t,CY} = \alpha_{x,t}E_{x,1,CY} \tag{9}$$

$$E_{x+t,1-t,CY} = \alpha_{x+t,1-t}E_{x,1,CY} \tag{10}$$

That is, the partial age exposure for a cohort year is the exposure for the full year of age multiplied by the partial age exposure weight. (9) and (10) allow the partial age cohorts at the start and end of the study to be expressed in terms for the full age exposure. Using the age 70 example above for the partial age cohorts 1941 ($CY = 1$) and 1944 ($CY = 4$), we would have:

$$E_{x,t,4} = \alpha_{x,t}E_{x,1,4} \tag{11}$$

$$E_{x+t,1-t,1} = \alpha_{x+t,1-t}E_{x,1,1} \tag{12}$$

Assuming cohorts have the same distribution, the errors in the rates for each cohort will be the same. So, for all cohorts, we would have:

$$\varepsilon_{x,t,CY} = \varepsilon_{x,t}\ , \tag{13}$$

$$\varepsilon_{x+t,1-t,CY} = \varepsilon_{x+t,1-t}\ . \tag{14}$$

Substituting (11), (12), (13), and (14) into (4) gives us:

$$\varepsilon_x = \left(\alpha_{x+t,1-t}E_{x,1,1}\varepsilon_{x+t,1-t} + \alpha_{x,t}E_{x,1,N+1}\varepsilon_{x,t}\right)/E_x \tag{15}$$

When the cohorts are equal in size and distribution, their exposures are also equal. Substituting $E_{x,1,1}$ for $E_{x,1,N+1}$in (15), we have:

$$\varepsilon_x = \left(\alpha_{x+t,1-t}\varepsilon_{x+t,1-t} + \alpha_{x,t}\varepsilon_{x,t}\right)E_{x,1,1}/E_x \tag{16}$$

Using §3.3 (16), which states that $\alpha_{x,t}\varepsilon_{x,t} = -\alpha_{x+t,t}\varepsilon_{x+t,1-t}$, we find that, if cohorts have the same distribution and size, then the total error in the study will be equal to zero:

$$\varepsilon_x = 0 \tag{17}$$

In a full study, the required cohorts would be defined by the minimum and maximum ages in the study given the study period. For a study from ages 20 to 90 for the study period 2012-2014, the youngest cohort contributing to age 20 would be lives born in 1994, while the oldest cohort contributing to age 90 would be 1921. To calculate the rates from age 20 to age 90, such that the partial-age errors offset to zero as in (17), would require the cohorts shown in the table below.

Age	Cohorts
20	1991-1994
21	1990-1993
…	
70	1941-1944
…	
90	1921-1924

Note that cohort years 1992 to 1994 could also contribute to ages 17 to 19, but these are below the minimum age in the study and are therefore excluded. Similarly, cohort years 1921 to 1923 would contribute to ages 91 to 93 which are greater than the maximum age and are again excluded.

Where the cohorts are not equal in size, there will be some residual error related to the difference in size between the starting and ending cohorts for the full year of age. This will be investigated further in following sections.

§4.2 Exposure Weights

The result from §4.1 (17), that errors in the partial ages at the start and end of the study net to zero for cohorts equal in size, depends on two assumptions:

1. That the mortality distribution in the year of age is the same across all cohorts contributing to the age.
2. That the distribution of birthdays within each cohort **year is the same** across all cohorts.

In the following formulas, (1) through (8), it will be shown that aggregate (i.e., applicable to the overall study) partial age exposure weights can be calculated in terms of cohort exposures. We start by noting that total exposure is the sum of the exposures for all cohorts:

$$E_x = E_{x+t,1-t,1} + E_{x,1,2} + E_{x,1,3} + .. + E_{x,1,N} + E_{x,t,N+1} \tag{1}$$

Utilizing $\alpha_{x,t,CY} + \alpha_{x+t,1-t,CY} = 1$ for the full age exposures, we have:

$$E_x = E_{x+t,1-t,1} + \left(\alpha_{x,t,2} + \alpha_{x+t,1-t,2}\right)E_{x,1,2} + \left(\alpha_{x,t,3} + \alpha_{x+t,1-t,3}\right)E_{x,1,3} + \cdots + \left(\alpha_{x,t,N} + \alpha_{x+t,1-t,N}\right)E_{x,1,N} + E_{x,t,N+1} \tag{2}$$

Rearranging the terms in the formula above, we have:

$$E_x = E_{x+t,1-t,1} + \left(\alpha_{x+t,1-t,2}E_{x,1,2} + \cdots + \alpha_{x+t,1-t,N}E_{x,1,N}\right) + \left(\alpha_{x,t,2}E_{x,1,2} + \cdots + \alpha_{x,t,N}E_{x,1,N}\right) + E_{x,t,N+1} \tag{3}$$

Assuming that all cohorts have the same distribution, we can use §4.1 (7) and (8) to substitute aggregate exposure weights for cohort exposure weights, resulting in the following:

$$E_x = E_{x+t,1-t,1} + \alpha_{x+t,1-t}\left(E_{x,1,2} + \cdots + E_{x,1,N}\right) + \alpha_{x,t}\left(E_{x,1,2} + \ldots + E_{x,1,N}\right) + E_{x,t,N+1} \tag{4}$$

Comparing (3) and (4), we find that the aggregate exposure weights can be calculated as:

$$\alpha_{x,t} = \left(\Sigma_{CY=2}^{N} \alpha_{x,t,CY} E_{x,1,CY}\right) / \left(\Sigma_{CY=2}^{N} E_{x,1,CY}\right) \tag{5}$$

$$\alpha_{x+t,1-t} = \left(\Sigma_{CY=2}^{N} \alpha_{x+t,1-t,CY} E_{x+t,t,CY}\right) / \left(\Sigma_{CY=2}^{N} E_{x,1,CY}\right) \tag{6}$$

Substituting §4.1 (5) and 6) into the above two formulas, we find that the aggregate exposure weights can be calculated as a sum of partial year exposures divided by a sum of full year exposures, where each sum include cohorts 2 through N:

$$\alpha_{x,t} = \left(\Sigma_{CY=2}^{N} E_{x,t,CY}\right) / \left(\Sigma_{CY=2}^{N} E_{x,1,CY}\right) \tag{7}$$

$$\alpha_{x+t,1-t} = \left(\Sigma_{CY=2}^{N} E_{x+t,1-t,CY}\right) / \left(\Sigma_{CY=2}^{N} E_{x,1,CY}\right) \tag{8}$$

§4.3 Increasing Cohorts

For ages with $N+1$ equally-sized cohorts, errors will offset, as illustrated by age 70 in the preceding table and as shown by formula §4.1(10). If the cohorts are not equal in size, then the errors for these ages will only partially offset. The net errors will be dependent on the cohort distribution.

The previously developed formula for the net error for ages with $N+1$ cohorts, §4.1 (15), is:

$$\varepsilon_x = \left(\alpha_{x+t,1-t} E_{x,1,1} \varepsilon_{x+t,1-t} + \alpha_{x,t} E_{x,1,N+1} \varepsilon_{x,t}\right) / E_x \tag{1}$$

This error has a systematic bias that arises from the error in the study method interacting with the difference in cohort sizes. As such, the error can be estimated and, if required, eliminated.

Let I be the increase in size between the first and last cohorts, i.e.

$$I = E_{x,1,N+1} / E_{x,1,1} - 1 \tag{2}$$

Substituting (2) into (1) and rearranging gives:

$$\varepsilon_x = \left(\alpha_{x+t,1-t} \varepsilon_{x+t,1-t} + \alpha_{x,t}(1+I)\varepsilon_{x,t}\right) E_{x,1,1} / E_x \tag{3}$$

First substituting $-\alpha_{x+t,1-t}\varepsilon_{x+t,1-t}$ for $\alpha_{x,t}\varepsilon_{x,t}$, using §3.3 (16), and then substituting $E_{x,1,N+1}/E_{x,1,1} - 1$ for I, using (2) above, gives:

$$\varepsilon_x = -\,\varepsilon_{x+t,1-t}\left(\alpha_{x+t,1-t} E_{x,1,N+1} - \alpha_{x+t,1-t} E_{x,1,1}\right) / E_x \tag{4}$$

Next, substituting $E_{x+t,1-t,1}$ for $\alpha_{x+t,1-t} E_{x,1,1}$, using §4.1 (10), and then substituting $E_{x,t,N+1}/\alpha_{x,t}$ for $E_{x,1,N+1}$, using §4.1 (9), the above formula can be expressed in terms of the partial-age exposures for the first and last cohorts:

$$\varepsilon_x = -\,\varepsilon_{x+t,1-t}\left(\left(\alpha_{x+t,1-t}/\alpha_{x,t}\right) E_{x,t,N+1} - E_{x+t,1-t,1}\right) / E_x \tag{5}$$

Formula §3.3 (16) can be rearranged to substitute $-\,\varepsilon_{x,t}\left(\alpha_{x,t}/\alpha_{x+t,1-t}\right)$ for $\varepsilon_{x+t,1-t}$ in the above formula, to yield the following result:

$$\varepsilon_x = \varepsilon_{x,t}\big(E_{x,t,N+1} - (\alpha_{x,t}/\alpha_{x+t,1-t})E_{x+t,1-t,1}\big)/E_x \tag{6}$$

Note that formulas (5) and (6) are equivalent, alternative formulas. The above formula shows that the error in the study can be estimated from

- the error in the exposure method, $\varepsilon_{x,t}$ using (6), or $\varepsilon_{x+t,1-t}$ using (5),
- the partial age exposures, $E_{x,t,N+1}$ and $E_{x+t,1-t,1}$ and
- the partial age exposure weights, $\alpha_{x,t}$ and $\alpha_{x+t,1-t}$, which can be estimated using §4.2 (7) and (8).

That estimate can then be used to eliminate the bias in the rates calculated by the study.

Later, a more general error formula, §6.1 (1), will be developed to estimate the error in the exposure method. That, together with formulas (6) above and §4.2 (7) and (8), can be used to estimate the bias in the rate and correct for it by subtracting the estimated error from the rate.

§4.4 Increasing Cohort Model

To investigate the impact of increasing cohorts on the errors in a study, consider a study of N calendar years with cohort size increasing at a simple (i.e., not compounded) annual rate, i. We will assume that the age has $N+1$ cohorts and therefore that the increase in size between the earliest and latest cohort for the age is Ni. The error formula developed here will be used in §6.6 to examine the impact of a range of cohort increases on the errors in the study.

By definition:

$$Ni = E_{x,1,N+1}/E_{x,1,1} - 1 \tag{1}$$

Next, we modify §4.3 (3) by substituting $(1+Ni)$ for $(1+I)$:

$$\varepsilon_x = \big(\alpha_{x+t,1-t}\varepsilon_{x+t,1-t} + \alpha_{x,t}(1+Ni)\varepsilon_{x,t}\big)E_{x,1,1}/E_x \tag{2}$$

Applying §3.3 (16), $\alpha_{x,t}\varepsilon_{x,t} = -\alpha_{x+t,1-t}\varepsilon_{x+t,1-t}$, we find:

$$\varepsilon_x = -\,\alpha_{x+t,1-t}\varepsilon_{x+t,1-t}NiE_{x,1,1}/E_x \tag{3}$$

Total exposure can be expressed as the sum of the exposure for each cohort:

$$E_x = E_{x+t,1-t,1} + E_{x,1,2} + .. + E_{x,1,N} + E_{x,t,N+1} \tag{4}$$

Replacing each cohort's exposure with partial age exposure weights times the cohort's full exposure and utilizing $\alpha_{x,t} + \alpha_{x+t,1-t} = 1$, we have:

$$E_x = \alpha_{x+t,1-t}E_{x,1,1} + \big(\alpha_{x,t} + \alpha_{x+t,1-t}\big)E_{x,1,2} + \ldots + \big(\alpha_{x,t} + \alpha_{x+t,1-t}\big)E_{x,1,N} + \alpha_{x,t}E_{x,1,N+1} \tag{5}$$

Grouping the above exposures by exposure weight gives:

$$E_x = \alpha_{x+t,1-t}\big(E_{x,1,1} + E_{x,1,2} + \cdots + E_{x,1,N}\big) + \alpha_{x,t}\big(E_{x,1,2} + \ldots + E_{x,1,N} + E_{x,1,N+1}\big) \tag{6}$$

Exposure for cohort K can be calculated in terms of $E_{x,1,1}$ as $(1+(K-1)i)E_{x,1,1}$. Substituting this expression for the exposures in (6) yields:

$$E_x = \alpha_{x+t,1-t}\left(E_{x,1,1} + (1+i)E_{x,1,1} + \cdots + (1+(N-1)i)E_{x,1,1}\right) + \alpha_{x,t}\left((1+i)E_{x,1,1} + \cdots + (1+(N-1)i)E_{x,1,1} + (1+Ni)E_{x,1,1}\right) \quad (7)$$

Using the arithmetic sum of the increases and rearranging gives:

$$E_x = \alpha_{x+t,1-t}E_{x,1,1}(N + ½(N-1)Ni) + \alpha_{x,t}E_{x,1,1}(N + ½N(N+1)i) \quad (8)$$

Replacing $\alpha_{x,t}$ with $1 - \alpha_{x+t,1-t}$ and simplifying further expresses total exposure in terms of the exposure for cohort 1, the study period, the annual increase, i, and the exposure weight, $\alpha_{x+t,1-t}$:

$$E_x = E_{x,1,1}\left(N + ½(N+1)Ni - \alpha_{x+t,1-t}Ni\right) \quad (9)$$

Substituting (9) into (3) gives:

$$\varepsilon_x = -\,\alpha_{x+t,1-t}\varepsilon_{x+t,1-t}Ni/\left(N + ½(N+1)Ni - \alpha_{x+t,1-t}Ni\right) \quad (10)$$

Once the error in the study methods has been established, the above formula will be used in §6.6 to investigate the impact of cohort increases on the error arising in a study.

§4.5 Cohort Periods

The cohorts in a study are sometimes grouped to study how mortality changes over time between groups of cohorts. For select rates, study cohorts are often defined by both year of birth and year of issue, where issue periods may reflect periods of different underwriting conditions. As the rates calculated for each age in a study are dependent on the blending of different cohorts to offset much of the errors from the study method, grouping by cohort period will create additional errors at the cohort group level, although no change in the error for all cohort groups combined. In this respect, the errors for sub-populations defined by cohort periods differ from those defined by other mortality risk variables, such as gender, where the different cohorts combine to offset much of the errors.

Consider the cohorts from 1941 to 1944 to be one such cohort group. The following table summarizes exposure by cohort year and age, showing the seven ages from 67 to 73 that receive contributions from the four cohorts. The boundary ages, 67 and 73, only receive contributions from one cohort, 1944 and 1941 respectively, while age 70 receives contributes from all 4 cohorts.

Exposure by Cohort Year and Age					
Age	1941	1942	1943	1944	Total
67				$E_{67+t,1-t,1}$	E_{67}
68			$E_{68+t,1-t,1}$	$E_{68,1,2}$	E_{68}
69		$E_{69+t,1-t,1}$	$E_{69,1,2}$	$E_{69,1,3}$	E_{69}
70	$E_{70+t,1-t,1}$	$E_{70,1,2}$	$E_{70,1,3}$	$E_{70,t,4}$	E_{70}
71	$E_{71,1,1}$	$E_{71,1,2}$	$E_{71,t,3}$		E_{71}
72	$E_{72,1,1}$	$E_{72,t,2}$			E_{72}
73	$E_{73,t,1}$				E_{73}

As before, we will use M to designate the number of cohorts which contribute exposure to age x. In a study of N calendar years, M can vary from 1 to the lesser of $N + 1$ and the total number of years in the cohort period, where the cohort period has been selected from the cohort range determined by the minimum and maximum ages in the study. In the table above, age 70 has contributions from 4 cohorts, while ages 67 and 73 have contributions from 1 cohort. The parallelogram formed by the exposure cells in the above table is representative of any cohort period that falls more than N years from the end of the cohort range defined by the minimum and maximum ages in the study. In this example, with the age range 20 to 90 for study years 2011 to 2014, that translates to any cohort period falling within the years 1925 and 1990. For cohort periods that start or end at the boundaries of study cohort range, the upper and lower triangles in the parallelogram will become truncated.

To illustrate an upper triangle, consider the cohort period from 1991 to 1994 with minimum age 20.

Exposure by Cohort Year and Age					
Age	1991	1992	1993	1994	Total
20	$E_{20+t,1-t,1}$	$E_{20,1,2}$	$E_{20,1,3}$	$E_{20,t,4}$	E_{20}
21	$E_{21,1,1}$	$E_{21,1,2}$	$E_{21,t,3}$		E_{21}
22	$E_{22,1,1}$	$E_{22,t,2}$			E_{22}
23	$E_{23,t,1}$				E_{23}

The cohort period from 1921 to 1924 with maximum age 90 illustrates a lower triangle:

Exposure by Cohort Year and Age					
Age	1921	1922	1923	1924	Total
87				$E_{87+t,1-t,1}$	E_{87}
88			$E_{88+t,1-t,1}$	$E_{88,1,2}$	E_{88}
89		$E_{89+t,1-t,1}$	$E_{89,1,2}$	$E_{89,1,3}$	E_{89}
90	$E_{90+t,1-t,1}$	$E_{90,1,2}$	$E_{90,1,3}$	$E_{90,t,4}$	E_{90}

The next table shows rate errors by cohort year and age for the cohort period 1941 to 1944.

Rate Errors by Cohort Year and Age					
Age	1941	1942	1943	1944	Total
67				$\varepsilon_{67+t,1-t,4}$	ε_{67}
68			$\varepsilon_{68+t,1-t,3}$	0	ε_{68}
69		$\varepsilon_{69+t,1-t,2}$	0	0	ε_{69}
70	$\varepsilon_{70+t,1-t,1}$	0	0	$\varepsilon_{70,t,4}$	ε_{70}
71	0	0	$\varepsilon_{71,t,3}$		ε_{71}
72	0	$\varepsilon_{72,t,2}$			ε_{72}
73	$\varepsilon_{73,t,1}$				ε_{73}

The above table illustrates three different patterns of partial ages that contribute to errors:

1. Only a cohort's first year of exposure contributes a partial age of exposure (ages 67 to 69).
2. Both the cohort's first and last year of exposure contribute a partial age of exposure (age 70).
3. Only a cohort's last year of exposure contributes a partial age of exposure (ages 71 to 73).

For the youngest and oldest ages, only one cohort contributes to its partial-age errors. For the youngest ages, 67 to 69, the total error for each age is:

$$\varepsilon_x = \left(E_{x+t,1-t,1}/E_x\right)\varepsilon_{x+t,1-t,1} \quad (1)$$

For the oldest ages, 71 to 73, the total error for each age is:

$$\varepsilon_x = \left(E_{x,t,M}/E_x\right)\varepsilon_{x,t,M} \quad (2)$$

For the middle ages, which is only age 70 in this example, the full formula shown in §4.1 (4) applies and is equal to the sum of the above two parts.

In summary, the total errors for each age can be calculated in terms of the method errors for the youngest and/or oldest ages, with the size of each error being proportional to its exposure.

Let $f_{x,CY}$ designate the length of time that age x is exposed for cohort year CY. For age 70, the sum of these partial age periods is equal to the number of calendar years in the study period, $N = 3$ calendar years, while for the three younger ages and the three older ages, this sum is less than the study period N. The farther the age is from age 70, the fewer years of exposure it contains. The following table summarizes the number of cohort years at each age together with the total years exposed.

Total Years Exposed		
Age	M	$\Sigma_{CY=1}^{M} f_{x,CY}$
67	1	$1-t$
68	2	$2-t$
69	3	$3-t$
70	4	3
71	3	$2+t$
72	2	$1+t$
73	1	t

Next, we will calculate errors by age compared to total exposure for the age, which we will refer to as the **total rate error**. This captures the overall distortion of a rate due to the error in its total exposure. We will employ two simplifying assumptions:

1. Exposure will be assumed to be uniformly distributed over the year of age.
2. Birthdays will be assumed to be uniformly distributed over the calendar year.

To understand the following formulas and the following table, it is helpful to remember that the partial ages that create errors have only half as much exposure as each full year of age which contributes no error. For age 68, with 1½ years of exposure, this means that total exposure is three times the partial age's half year of exposure contributed by its 1943 cohort. Hence, the partial age error is divided by 3 to calculate the error compared to total exposure, i.e., the total rate error.

Using M to designate the number of cohort years that contribute exposure to the age, the error for the youngest ages with fewer than $\mathrm{N}+1$ cohort years, such as ages 67 to 69 in the example above, can be approximated by:

Total rate error for youngest ages: $\varepsilon_x \approx \varepsilon_{x+½,½,1}/(2M-1)$ (3)

Similarly, the error for the oldest ages with fewer than $\mathrm{N}+1$ cohort years, such as ages 71 to 73 in the example above, can be approximated by:

Total rate error for oldest ages: $\varepsilon_x \approx \varepsilon_{x,½,M}/(2M-1)$ (4)

The following table illustrates the results of applying the above formulas for total rate error to the youngest and oldest ages in the study previously illustrated:

Rate Errors by Cohort Year and Age					
Age	1941	1942	1943	1944	Total Rate Error
67				$\varepsilon_{67+½,½,1}$	$\varepsilon_{67} = \varepsilon_{67+½,½,1}$
68			$\varepsilon_{68+½,½,1}$	0	$\varepsilon_{68} = \varepsilon_{68+½,½,1}/3$
69		$\varepsilon_{69+½,½,1}$	0	0	$\varepsilon_{69} = \varepsilon_{69+½,½,1}/5$
70	$\varepsilon_{70+½,½,1}$	0	0	$\varepsilon_{70,½,4}$	$\varepsilon_{70} = 0$
71	0	0	$\varepsilon_{71,½,3}$		$\varepsilon_{71} = \varepsilon_{71,½,3}/5$
72	0	$\varepsilon_{72,½,2}$			$\varepsilon_{72} = \varepsilon_{72,½,2}/3$
73	$\varepsilon_{73,½,1}$				$\varepsilon_{73} = \varepsilon_{73,½,1}$

In the table above, only one age, age 70, has exposure from $N+1$ cohorts and, therefore, has partial age errors that offset. The total rate error becomes an increasing proportion of total exposure as the lowest ages approach the minimum age or the highest ages approach the maximum age. Generally, the number of youngest and oldest ages with non-offsetting errors will be equal to the lesser of the number of cohorts and the number of years in the study period. As mortality generally increases with age, the lower age errors will usually be positive and the upper age errors will usually be negative but larger in magnitude, due to higher rates at higher ages. Examining the overall error for the cohort group across all ages, the lowest and highest age errors will offset but, as the errors for the highest ages are larger, the overall error will be negative and hence the overall mortality rate will understate the true overall mortality rate for the cohort group.

§4.6 Fractional Methods

Unlike the annual exposure study methods that prorate annual exposure for partial ages, fractional methods use fractional exposure which allocates an exposure of 1 to lives active for the entire fractional period. Fractional rates or forces are calculated for each fractional period in the year of age as fractional period deaths divided by fractional exposure.

Fractional survival rates are calculated as one minus fractional mortality rates. An annual survival rate is then calculated as the product of the fractional survival rates. The annual mortality rate is finally calculated as one minus the annual survival rate, as shown below:

$$q_x = 1 - \Pi_1^P\left(1 - {}_f q_{x+s}\right)$$

The annual force is calculated more simply as the sum of the fractional forces:

$$\bar{\mu}_x = \Sigma_1^P {}_f\bar{\mu}_{x+s}$$

For simplicity, consider a half-yearly study for calendar years 2012 to 2014 with four half-year cohorts for lives born in the first and second halves of 1943 and 1944. We will further assume that policies were only issued at the half year, either January 1 or July 1, using a half-year anniversary definition for both age and cohorts. This will enable deaths, exposure and rates to be calculated for half-year ages and half-year cohorts, as illustrated in the following table:

Exposure by Half-Year Cohorts and Ages

Age	1943,1	1943,2	1944,1	1944,2	Total
67½				${}_{½}E_{67½,4}$	${}_{½}E_{67½}$
68			${}_{½}E_{68,3}$	${}_{½}E_{68,4}$	${}_{½}E_{68}$
68½		${}_{½}E_{68½,2}$	${}_{½}E_{68½,3}$	${}_{½}E_{68½,4}$	${}_{½}E_{68½}$
69	${}_{½}E_{69,1}$	${}_{½}E_{69,2}$	${}_{½}E_{69,3}$	${}_{½}E_{69,4}$	${}_{½}E_{69}$
69½	${}_{½}E_{69½,1}$	${}_{½}E_{69½,2}$	${}_{½}E_{69½,3}$	${}_{½}E_{69½,4}$	${}_{½}E_{69½}$
70	${}_{½}E_{70,1}$	${}_{½}E_{70,2}$	${}_{½}E_{70,3}$	${}_{½}E_{70,4}$	${}_{½}E_{70}$
70½	${}_{½}E_{70½,1}$	${}_{½}E_{70½,2}$	${}_{½}E_{70½,3}$		${}_{½}E_{70½}$
71	${}_{½}E_{71½,1}$	${}_{½}E_{71½,2}$			${}_{½}E_{71}$
71½	${}_{½}E_{71½,1}$				${}_{½}E_{71½}$

In the table above, all ages, except for the first and last, have two or more half years. For ages with two half-years of exposure, an annual rate could be calculated as

$$q_x = 1 - (1 - {}_{½}q_x)(1 - {}_{½}q_{x+½})$$

The annual rates calculated from the fractional rates are independent of the size of the cohorts, unlike the annual method for which cohorts of different sizes will introduce errors in the rates.

Where the fractional period is defined with sufficient granularity (i.e., short enough) such that partial fractional periods cannot occur or can be ignored, the fractional rates will not contain material errors due to partial fractional periods and hence the annual rates will be without material error.

Chapter 5: Actual Rate Distributions

The exposure calculated for a partial age in a study will reflect the mortality distribution of the actual deaths in the study irrespective of the mortality distribution assumed by the method used to calculate exposure. The difference between the actual distribution and the distribution implicit in the study method will result in errors when mortality rates are calculated based on data that includes partial ages. To explore and quantify the impact of errors in the rates from partial ages arising in calendar year studies, a model of the actual distribution of rates over the year is required. The rate distribution can be considered the series of fractional rates that would arise in a fractional study for a full year of age. If the year were divided into P fractional periods of length f, that series of fractional rates would be:

$${}_fq_x, {}_fq_{x+f}, {}_fq_{x+2f} \; \cdots, {}_fq_{x+(P-1)f}.$$

For the partial years of age that arise when a full year of age is split between two calendar years, we will assume that the calendar year-end falls at the end of a fractional period, so that each partial age is made up of an integer number of fractional periods. As both fractional and partial rates are being considered, the fractional age notation using P, s and f will be used for the fractional periods, while the partial age notation will use t to indicate the time from the start of the age in a calendar year to the end of the calendar year. The partial rate for each partial age is given by

$${}_tq_x = 1 - \Pi_{s=1}^{Pt}\left(1 - {}_fq_{x+s}\right) \tag{1}$$

$${}_{1-t}q_{x+t} = 1 - \Pi_{s=Pt+1}^{P}\left(1 - {}_fq_{x+s}\right) \tag{2}$$

The annualized rates for the partial ages, using the partial age notation above, are

$$q_{x,t} = 1 - (1 - {}_tq_x)^{1/t} = 1 - \left(\Pi_{s=1}^{Pt}\left(1 - {}_fq_{x+s}\right)\right)^{1/t} \tag{3}$$

$$q_{x+t,1-t} = 1 - (1 - {}_{1-t}q_{x+t})^{1/(1-t)} = 1 - \left(\Pi_{s=Pt+1}^{P}\left(1 - {}_fq_{x+s}\right)\right)^{1/(1-t)} \tag{4}$$

The formula is greatly simplified when using the force of mortality over the year, as the force of mortality is additive, while mortality rates are the complement of multiplicative survival rates. If the fractional forces over the year are ${}_f\bar{\mu}_x, {}_f\bar{\mu}_{x+f}, {}_f\bar{\mu}_{x+2f} \; \cdots, {}_f\bar{\mu}_{x+(P-1)f}$, then:

The partial force for each partial age is given by

$${}_t\bar{\mu}_x = \Sigma_{s=1}^{Pt}\, {}_f\bar{\mu}_{x+s} \tag{5}$$

$${}_{1-t}\bar{\mu}_{x+t} = \Sigma_{s=Pt+1}^{P}\, {}_f\bar{\mu}_{x+s} \tag{6}$$

The annualized forces for the partial ages are

$$\bar{\mu}_{x,t} = {}_t\bar{\mu}_x/t \tag{7}$$

$$\bar{\mu}_{x+t,1-t} = {}_{1-t}\bar{\mu}_{x+t}/(1-t) \tag{8}$$

As mortality is continuous and generally increasing from age to age, a simple model for the force distribution is that it increases linearly over the year. We will apply two constraints to the force distribution, referred to as consistency and continuity:

- Consistency: the sum of the fractional forces over the year must equal the annual force, and
- Continuity: the exact force at the end of the year of age must equal to the exact force at the start of the following year.

Since the mortality force curve is non-linear, a linear force distribution cannot satisfy both constraints simultaneously:

- If continuity is maintained, then annual force over the year will not equal the sum of the fractional forces.
- If consistency is maintained, the forces will not be continuous from age to age.

For the sections of the mortality force curve where the annual force is small and its rate of increase changes slowly from age to age, a linear force distribution can be used without seriously violating the either constraint. However, as the aim is to project lives and deaths for a given mortality rate, a **linear force distribution** will be set out in §5.3 that will preserve consistency. Where the force is large, or the rate of increase is changing rapidly, the simple, linear force distribution is too crude. In §5.4, this is resolved by splitting the age's force into two separate but not necessarily equal linear segments. This will be called the **bi-linear force distribution** here. It is constructed so that both consistency and continuity are maintained. §5.5 illustrates the linear and bi-linear distributions for five consecutive ages, using select and ultimate mortality rates from the Male Nonsmoker, Age Nearest Birthday, 2015 Valuation Basic Table, which, going forward, will be referred to as "**MNS ANB 2015 VBT**."

§5.1 Increase in Force

The increase in force over the year of age x, $\Delta\mu_x$, is the difference between the instantaneous forces at the start and end of the year.

$$\Delta\mu_x = \mu_{x+1} - \mu_x \tag{1}$$

The increase in force can be interpreted as the average slope or gradient of the force of mortality curve between exact ages x and $x+1$.

The standard estimate of the instantaneous force of mortality at exact age x, μ_x, is the average of the annual forces for the current and prior ages.

$$\mu_x \approx \tfrac{1}{2}(\bar{\mu}_{x-1} + \bar{\mu}_x) \tag{2}$$

Applying (2) to both μ_{x+1} and μ_x, the increase in force can be restated as half the difference between the annual force for the next age, $\bar{\mu}_{x+1}$, and the average force for the previous age, $\bar{\mu}_{x-1}$:

$$\Delta\mu_x \approx \mu_{x+1} - \mu_x = \tfrac{1}{2}(\bar{\mu}_{x+1} - \bar{\mu}_{x-1}) \tag{3}$$

The relative increase in force, i.e. the increase in force divided by the annual force, is a useful measure to compare increases across a wide range of ages. It also simplifies some soon-to-be-developed formulas. Going forward, we will refer to the relative increase in force as the **relative gradient of the force**, the **relative gradient**, or simply the **gradient**. The relative gradient, Δ_x, not to be confused with $\Delta\mu_x$, is the increase in force for age x divided by the annual force for age x:

$$\Delta_x = \Delta\mu_x / \bar{\mu}_x \tag{4}$$

Since the minimum age, such as age 0, does not have a prior age, μ_0 and $\Delta\mu_0$cannot be calculated. The simplest remedy is to extrapolate the instantaneous force to age 0 by assuming the average force over age 0 is equal to the instantaneous force at age ½, i.e., $\bar{\mu}_0 = \mu_{½}$. Further assuming that the increase in instantaneous force from age 0 to age 1, $\mu_1 - \mu_0$, is twice the increase in instantaneous force from age ½ to age 1, we have:

$$\mu_1 - \mu_0 \approx 2(\mu_1 - \mu_{½}) = 2(\mu_1 - \bar{\mu}_0) \tag{5}$$

Solving (5) for μ_0, we have:

$$\mu_0 = 2\bar{\mu}_0 - \mu_1 \tag{6}$$

Using (2) to substitute for μ_1, we have:

$$\mu_0 = 2\bar{\mu}_0 - ½(\bar{\mu}_0 + \bar{\mu}_1) = ½(3\bar{\mu}_0 - \bar{\mu}_1) \tag{7}$$

The above approach may not be consistent with the trend in the relative gradients that follow the minimum age, such as Δ_1, Δ_2, and Δ_3. If the trend is relatively stable, a good alternative may be to set the relative gradient for the boundary age equal to the relative gradient for the following age, i.e., $\Delta_0 = \Delta_1$. Assuming these two relative gradients are equal and substituting for Δ_0 using (4) and rearranging, we find that the increase in force for age 0 is equal to the average force for age 0 multiplied by the relative gradient for age 1:

$$\Delta\mu_0 = \bar{\mu}_0\Delta_1 \tag{8}$$

However, if the trend in relative gradients is increasing or decreasing, a better approach may be to extrapolate the relative gradient for the boundary age from the relative gradients for the two following ages. Assuming that the ratios of successive relative gradients, Δ_0/Δ_1 and Δ_1/Δ_2, are equal, we can solve for Δ_0 in terms of Δ_1 and Δ_2:

$$\Delta_0 = (\Delta_1)^2/\Delta_2 \tag{9}$$

Substituting for Δ_0 using (4) and rearranging, we see that the increase in force for age 0 is equal to the average force at age 0 multiplied by the relative gradient for age 1 and the ratio of the relative gradients for ages 1 and 2.

$$\Delta\mu_0 = \bar{\mu}_0\Delta_0 = \bar{\mu}_0\Delta_1(\Delta_1/\Delta_2) \tag{10}$$

The relative gradients for the ultimate rates shown in the following section did not require extrapolation since the starting age of 50 was higher than the minimum age in the MNS ANB 2015 VBT table. However, formula (10) was used for first year select rates for all issue ages. Specifically, the relative gradient for issue age x in the first policy year was calculated as:

$$\Delta_{[x],1} = \Delta_{[x],2}/\left(\Delta_{[x],3}/\Delta_{[x],2}\right) \tag{11}$$

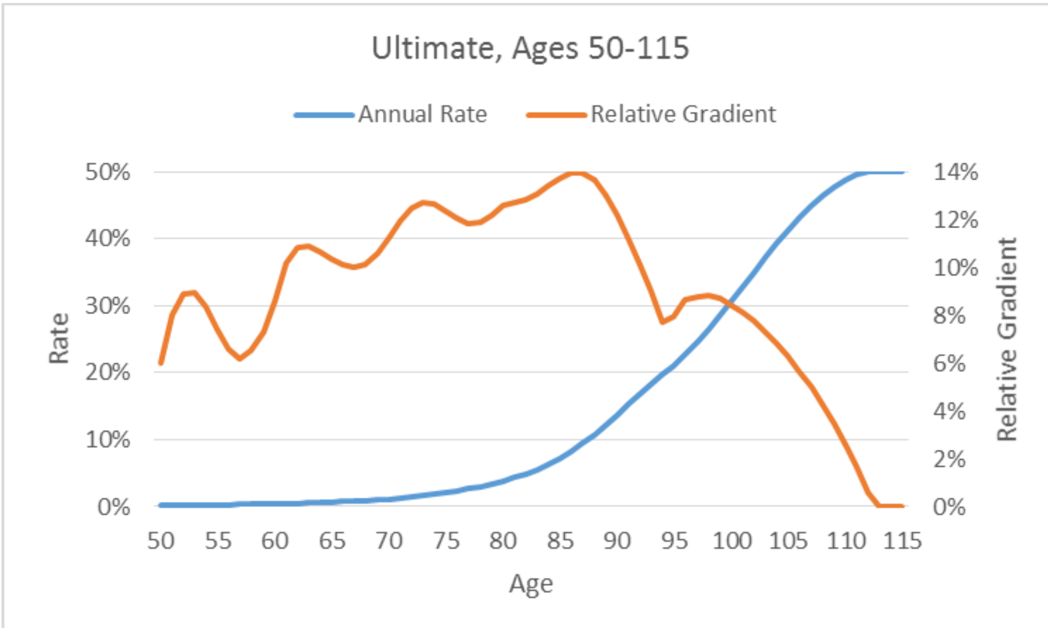
The increase in force for the first policy year was then calculated using (10) as:

$$\Delta\mu_{[x],1} = \bar{\mu}_{[x],1}\Delta_{[x],1} = \bar{\mu}_{[x],1}\Delta_{[x],2}\left(\Delta_{[x],2}/\Delta_{[x],3}\right) \tag{12}$$

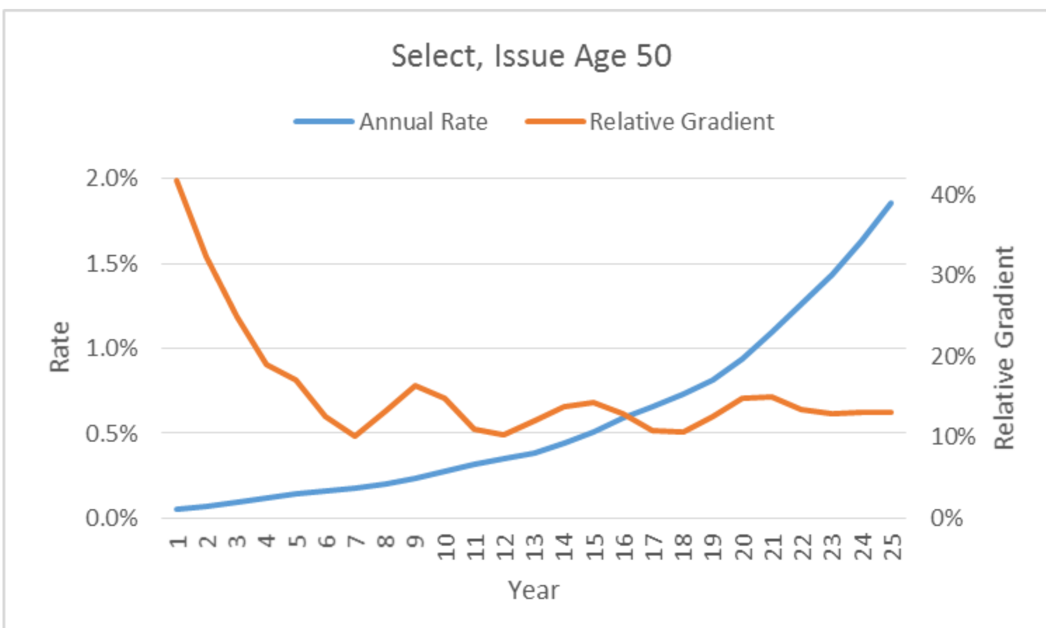
§5.2 Sample Table Relative Gradients

To illustrate the range of relative gradients that may arise in a study, the relative gradients are calculated across the age ranges using mortality rates from MNS ANB 2015 VBT. While ultimate rates are defined by attained age, select rates are defined by issue age and policy year, although this is equivalent to issue age and attained age. For simplicity here, "age" refers to attained age in either an ultimate or select table, although the select rate charts presented will continue to use policy year.

For ultimate mortality above age 50, the rates increase with age. The relative gradient increases from 6% to 14% between ages 50 and 87, oscillating about the trend, and then decreases to 0% at age 112 when the highest rate of 50% is reached, although there is a "kink" between ages 95 and 98 where the relative gradient increases. While such oscillations and kinks may seem odd, keep in mind that the relative gradient, equal to the increase in force divided by the force, is a new metric has never been used when finalizing mortality rates, although it may be of future interest.



For select mortality, there is a steeply decreasing relative gradient curve in the first 5 to 10 policy years leveling off around 10%. The relative gradient curve for issue age 50 is shown below.



The steep decline of the relative gradient curve in the first policy year increases with issue age, as the table below shows.

Issue Age	Year 1 Gradient
[50]	42%
[70]	61%
[90]	125%

§5.3 Linear Force Distribution

The linear force distribution assumes that the force changes linearly over the year of age, with the average force, $\bar{\mu}_x$, occurring halfway through the year of age at $x + ½$. This ensures consistency with the annual rate, as the sum of the force over the year will equal the average annual force, thereby reproducing the annual rate. The linear force distribution is a simple model for the actual distribution of deaths in the study. For clarity, linear force variables will use an LF superscript.

The linear force distribution reproduces $\bar{\mu}_x$ at age $x + ½$ and has a slope of $\Delta\mu_x$, which is the difference between the instantaneous forces at exact ages $x + 1$ and x:

$$\mu_{x+t}^{LF} = \bar{\mu}_x + (t - ½)\Delta\mu_x \tag{1}$$

Because the line is centered at $x + ½$, the average force for the year is equal to the instantaneous force at $x + ½$,:

$$\bar{\mu}_x^{LF} = \mu_{x+½}^{LF} = \bar{\mu}_x \tag{2}$$

The linear force distribution is not continuous from one age to the next: There are discontinuities at the end of each age. The instantaneous force at exact age x is no longer well defined, with different values at the start of age x, with $t = 0$, and at the end of age $x - 1$, with $t = 1$. Using §5.1(3) to substitute for $\Delta\mu_x$ in (3) and for $\Delta\mu_{x-1}$ in (4), we have:

$$\mu_{x+0}^{LF} = \bar{\mu}_x + (0 - ½)\Delta\mu_x = \bar{\mu}_x - ¼(\bar{\mu}_{x+1} - \bar{\mu}_{x-1}) \neq \mu_x \tag{3}$$

and

$$\mu_{(x-1)+1}^{LF} = \bar{\mu}_{x-1} + (1 - ½)\Delta\mu_{x-1} = \bar{\mu}_{x-1} + ¼(\bar{\mu}_x - \bar{\mu}_{x-2}) \neq \mu_x \tag{4}$$

The above formulas show that, except for occasional coincidences, the linear force at the start of age x will not equal the linear force at the end of age $x - 1$, nor will either equal the instantaneous force at age x, μ_x.

To apply the linear force distribution to a partial age, the average force for a fractional period, f, is required. The average force from exact age $x + s$ to $x + s + f$ is given by:

$${}_f\bar{\mu}_{x+s}^{LF} = ½f\left(\mu_{x+s}^{LF} + \mu_{x+s+f}^{LF}\right)$$

$$= ½f(\bar{\mu}_x + (s - ½)\Delta\mu_x + \bar{\mu}_x + (s + f - ½)\Delta\mu_x)$$

$$= f(\bar{\mu}_x + (s - ½(1 - f))\Delta\mu_x) \tag{5}$$

We will define $T_{s,f}$ define as the time from the middle of age x, i.e. $x + ½$, to the middle of the partial age, i.e., $x + s + ½f$. The result is:

$$T_{s,f} = s - ½(1 - f) \tag{6}$$

Substituting (6) into (5) and gives:

$$_f\bar{\mu}^{LF}_{x+s} = f\left(\bar{\mu}_x + T_{s,f}\Delta\mu_x\right) \tag{7}$$

The annual force for a partial age is equal to its annualized force. Using the partial age notation introduced above, the annual force for a partial age from exact age $x + s$ to $x + s + f$ is given by:

$$\bar{\mu}^{LF}_{x+s,f} = {}_f\bar{\mu}^{LF}_{x+s}/f = \bar{\mu}_x + T_{s,f}\Delta\mu_x \tag{8}$$

By factoring out $\bar{\mu}_x$, the above forces can be expressed in terms of the relative gradient, which was defined in §5.1(4):

$$\mu^{LF}_{x+t} = \bar{\mu}_x(1 + (t - ½)\Delta_x) \tag{9}$$

$$_f\bar{\mu}^{LF}_{x+s} = f\bar{\mu}_x\left(1 + T_{s,f}\Delta_x\right) \tag{10}$$

$$\bar{\mu}^{LF}_{x+s,f} = \bar{\mu}_x\left(1 + T_{s,f}\Delta_x\right) \tag{11}$$

§5.4 Bi-Linear Force Distribution

The linear force distribution reproduced the annual force for each age, thereby achieving consistency, but it lacked enough degrees of freedom to also achieve continuity between ages. In contrast, the bi-linear force distribution uses two linear segments within each age to achieve both consistency and continuity, as follows:

1. The first linear segment runs from μ_x, the instantaneous force at the start of the year of age x, to $\bar{\mu}_x$, the average force for age x, at a time to be solved for, which we will denote as $T_{\bar{\mu}_x}$.
2. The second linear segment runs from $\bar{\mu}_x$, the average force for age x, at time $T_{\bar{\mu}_x}$ to μ_{x+1}, the instantaneous force at the end of the year of age x.
3. By ending one age with the instantaneous force and starting the next age with the same instantaneous force, continuity is achieved.
4. Time $T_{\bar{\mu}_x}$ will be solved for such that the sum of the average forces for each segment, equals $\bar{\mu}_x$, thus insuring consistency is achieved.

Setting the sum of the time-weighted average force for each segment equal to the average force for age x, $\bar{\mu}_x$, yields the following:

$$½(\mu_x + \bar{\mu}_x)T_{\bar{\mu}_x} + ½(\bar{\mu}_x + \mu_{x+1})(1 - T_{\bar{\mu}_x}) = \bar{\mu}_x\,. \tag{1}$$

Solving for $T_{\bar{\mu}_x}$ yields:

$$T_{\bar{\mu}_x} = (\mu_{x+1} - \bar{\mu}_x)/(\mu_{x+1} - \mu_x)\,. \tag{2}$$

Substituting for instantaneous forces as the average of the annual force for the current and prior ages gives

$$T_{\bar{\mu}_x} = (\bar{\mu}_{x+1} - \bar{\mu}_x)/(\bar{\mu}_{x+1} - \bar{\mu}_{x-1}). \tag{3}$$

Extending the increase in force formula to partial ages, the partial force from exact age $x + s$ to $x + s + f$ is given by

$$\Delta_f \mu_{x+s} = f\left(\mu_{x+s+f} - \mu_{x+s}\right) \quad (4)$$

Applying (4) to the partial ages from exact age x to $x + T_{\bar{\mu}_x}$, and from $x + T_{\bar{\mu}_x}$ to $x + 1$, the increase in the partial forces for each segment are:

$$\Delta_{T_{\bar{\mu}_x}} \mu_x = \left(\mu_{x+T_{\bar{\mu}_x}} - \mu_x\right)/T_{\bar{\mu}_x} = ½\,(\bar{\mu}_x - \bar{\mu}_{x-1})/T_{\bar{\mu}_x} \quad (5)$$

$$\Delta_{1-T_{\bar{\mu}_x}} \mu_{x+T_{\bar{\mu}_x}} = \left(\mu_{x+1} - \mu_{x+T_{\bar{\mu}_x}}\right)/\left(1 - T_{\bar{\mu}_x}\right) = ½(\bar{\mu}_{x+1} - \bar{\mu}_x)/\left(1 - T_{\bar{\mu}_x}\right) \quad (6)$$

The exact linear force distribution is then given by

$$\mathrm{t} < T_{\bar{\mu}_x}: \mu_{x+t}^{BLF} = \bar{\mu}_x + \left(t - T_{\bar{\mu}_x}\right)\Delta_{T_{\bar{\mu}_x}} \mu_x \quad (7)$$

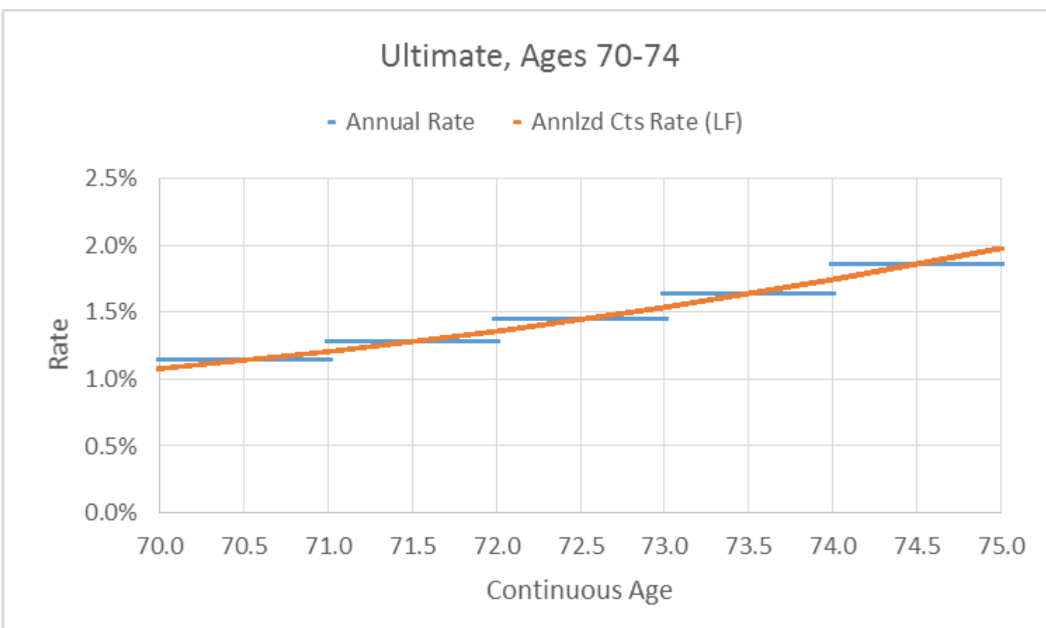
$$\mathrm{t} > T_{\bar{\mu}_x}: \mu_{x+t}^{BLF} = \bar{\mu}_x + \left(t - T_{\bar{\mu}_x}\right)\Delta_{1-T_{\bar{\mu}_x}} \mu_{x+T_{\bar{\mu}_x}} \quad (8)$$

Formulas (7) and (8) are used to project the forces over the year for the bi-linear force distribution.

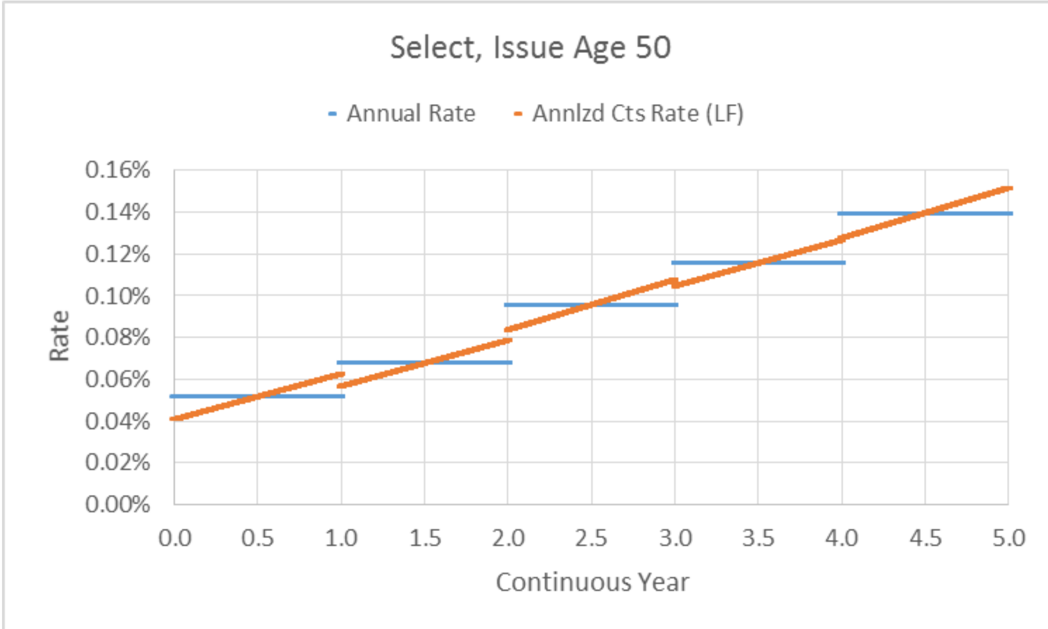
§5.5 Sample Linear Force Distributions

The distribution of rates both within and across rate years is illustrated below using five years of mortality rates for sample age ranges from the MNS ANB 2015 VBT. The graphs below show annualized rates calculated from linear forces in red and the original rates, which are level for each age or year, in blue.

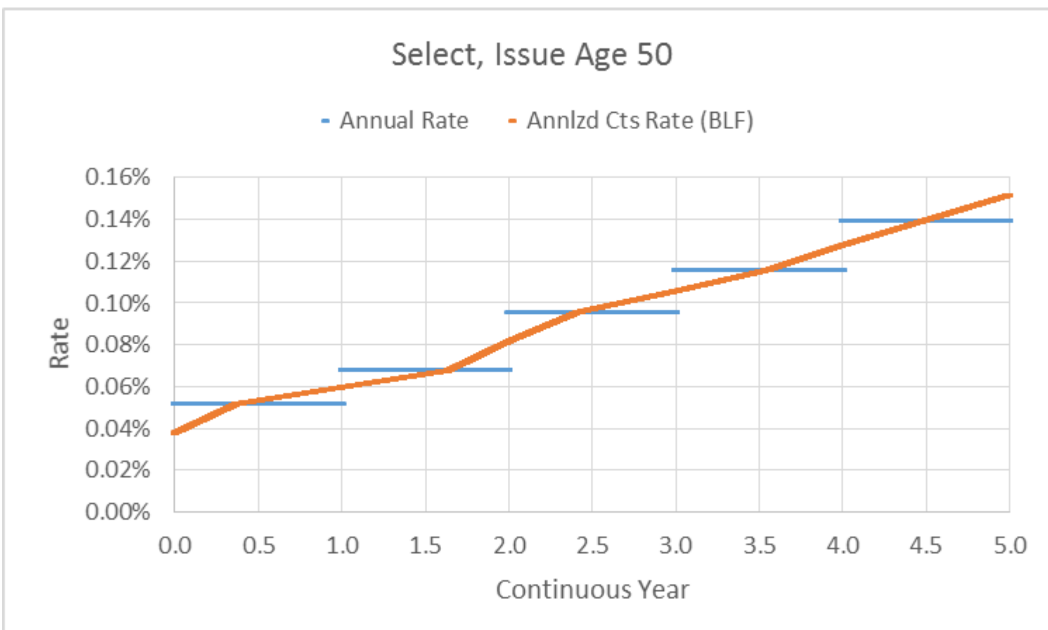
The first graph shows ultimate mortality rates. The blue lines show typical annual mortality rates which form a step function. The red mortality line, based on the linear force distribution, looks continuous because the discontinuities at these ages are so small that they are not visible. At some ages, such as 30, the discontinuities are more visible.



The next graph shows select mortality for the first five select years for issue age 50. Again, the blue lines show mortality rates as a step function. The red line shows annualized mortality rates based on the linear force distribution, this time with discontinuities that are clearly visible.



The next graph is similar to the previous graph, but with the red line using the bi-linear force distribution instead of the linear force distribution. Two linear segments per age are quite visible for policy years 1 to 3. Notice how the segments do not bisect the age. This is needed to reproduce the overall average force and mortality rate for each policy year.



Chapter 6: Partial Age Errors

The idea of a generalized error formula was sparked by examining partial age error patterns arising from sensitivity testing of the projections. This section will explore partial age errors and develop a generalized error formula.

The error in a partial age will first be estimated by using the linear force distribution as a proxy for the underlying, but unknown, distribution of deaths in a study. The linear force distribution is a simple model that reproduces the mortality rate, and is symmetric about the mid-point of the age. This model is robust enough for many situations, but not for the high relative gradients experienced for early-duration select rates at older ages. The linear force distribution is a useful starting point to identify the key drivers of partial age errors.

The first subsection, §6.1, will define and explore a generalized formula for partial age errors associated with each of three exposure methods. It will also examine the mortality distribution that is implicit in each exposure method. Next, §6.2 will examine the linear force distribution of deaths over the year of age compared to the distributions implicit in the three exposure methods. §6.3 will first apply error formulas to mortality rates for various age ranges from MNS ANB 2015 VBT. Next, annual rates for partial ages, and hence their errors, will be calculated directly as part of a projection.

§6.4 will illustrate partial age rates based on the bi-linear force distribution. The bi-linear force distribution will be used to more accurately model the distribution of deaths for early-duration select rates at the older ages, using MNS ANB 2015 VBT. This will lead to the generalized error formula being tested with the bi-linear force distribution.

§6.5 and §6.6 will accumulate partial age errors across cohorts in a study to examine the hybrid exposure method discussed in Chapter 3, as well as the impact of increasing cohorts. The hybrid exposure method will reduce partial age errors at the end of the study period but, as this error will not be symmetrical with the error at the start of the study, the overall errors in the study will be increased. In contrast, it was shown in §4.1 that the partial age errors across cohorts will offset to zero if the cohorts are equal in size. Formulas from §4.4 and §6.1 will be used to examine the partial age errors for a study composed of increasing cohorts.

Finally, the generalized error formula will be derived from first principles using fractional rates in §6.7.

§6.1 Generalized Error Formula

Partial age errors can be estimated using the following generalized error formula. A superscript of "G"is used to indicate a generalized formula that applies to multiple exposure methods. The following generalized error formula, which will be derived in §6.7, ignores cubic and higher order terms and is therefore approximate:

$$\varepsilon^{G}_{x+s,f} \approx T_{s,f}(\Delta_x + Mq_x)q_x \,, \tag{1}$$

where M takes on one of the following three values:

- $M = 0$ for the annual force method,
- $M = 1$ for the annual rate method with traditional exposure, and
- $M = -1$ for the annual rate method with distributed exposure.

Using $M = 0$, the error formula for the constant force method is:

$$\varepsilon_{x+s,f}^{CF} \approx T_{s,f} \Delta_x q_x \,. \tag{2}$$

Using $M = 1$, the error formula for the traditional exposure method is:

$$\varepsilon_{x+s,f}^{BH} \approx T_{s,f} (\Delta_x + q_x) q_x \,. \tag{3}$$

Using $M = -1$, the error formula for the distributed exposure method is:

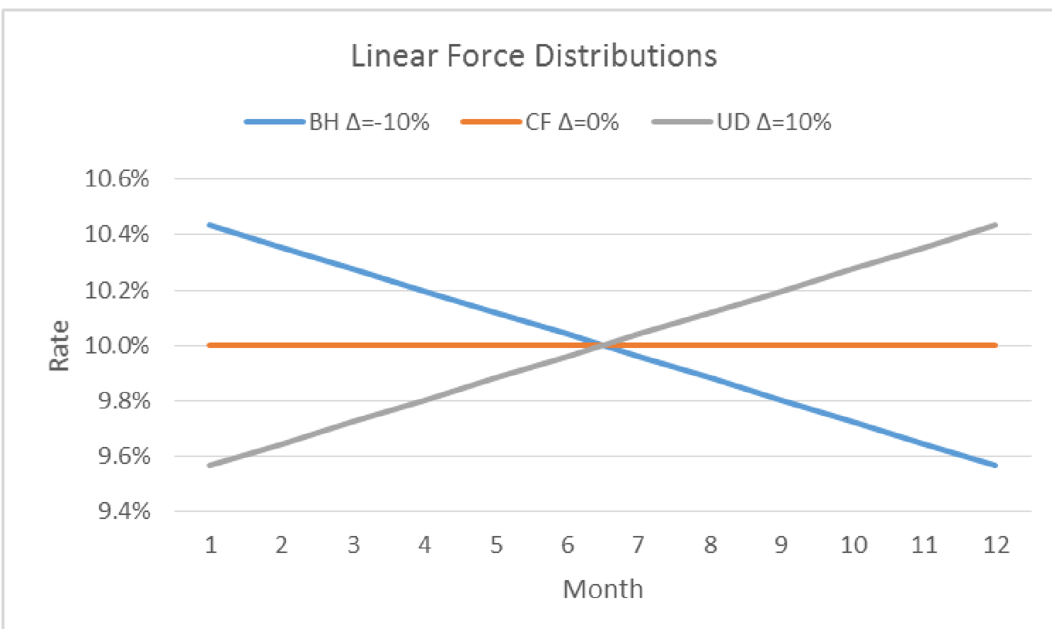
$$\varepsilon_{x+s,f}^{UD} \approx T_{s,f} (\Delta_x - q_x) q_x \,. \tag{4}$$

Setting the error formula to zero for each exposure method determines the relative gradients, and hence the linear force distribution, that will produce no errors under that method for a given mortality rate. That is,

- $\Delta_x \approx -q_x$.: Annual rate method with traditional exposure,
- $\Delta_x \approx 0$: Annual force method, and
- $\Delta_x \approx q_x$: Annual rate method with distributed exposure.

The graph below shows the annualized monthly rates corresponding to a 10% annual rate for the three exposure methods. The underlying distributions for the exposure methods were generated using the linear force distribution with the following relative gradients:

- $\Delta_x = -10\%$ for the annual rate method with traditional exposure (BH for Balducci Hypothesis),
- $\Delta_x = 0\%$ for the annual force method (CF for Constant Force), and
- $\Delta_x = 10\%$ for the annual rate method with distributed exposure (UD for Uniform Distribution of Deaths).

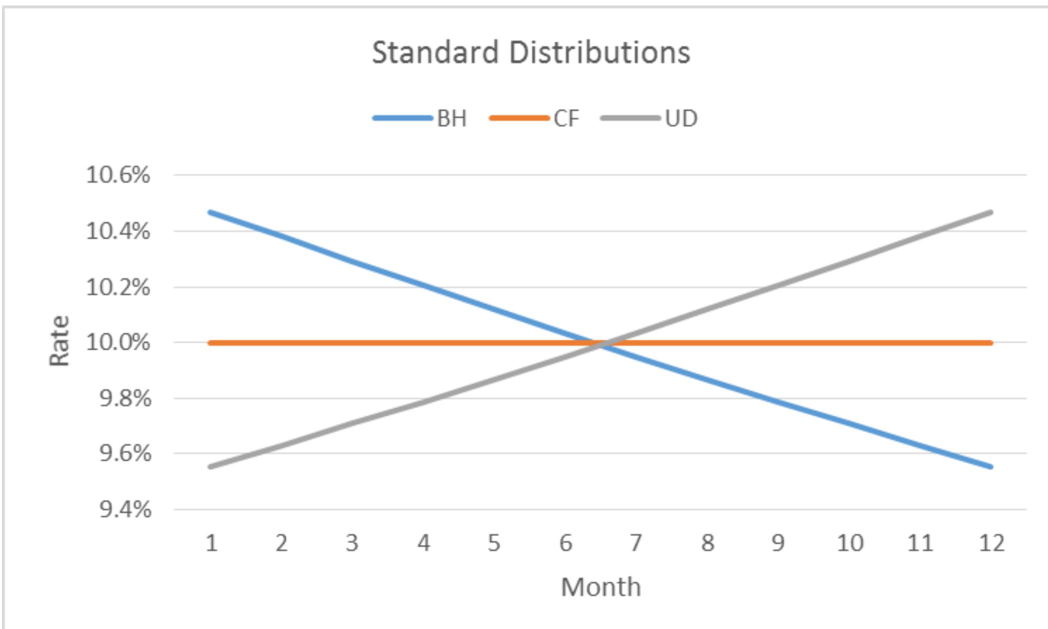


For the traditional method, the mortality distribution resulting from the Balducci hypothesis is approximately equal to the linear force distribution with a relative gradient equal to the mortality rate multiplied by minus one. For the distributed method, the mortality distribution resulting from the uniform distribution of deaths is approximately equal to the linear force distribution with a relative gradient equal to the mortality rate.

So, for both of the annual rate methods, the relative gradient and hence the linear force distribution that produces no error is dependent on the size of the mortality rate. For the distributed method, the relative gradient that minimizes errors is positive and hence the distribution of deaths is increasing over the year, while for the traditional method the relative gradient that minimizes errors is negative and hence the distribution of deaths is decreasing over the year.

For the annual force method, the distribution of deaths resulting from the constant force distribution is approximately equal to the linear force distribution with a relative gradient equal to zero, so is therefore independent of the mortality rate.

This compares favorably with the mortality distributions for each exposure method calculated using the distributions originally shown in §3.2 and reproduced below.



As mortality rates increase by age, the linear force distribution that produces no errors for each method will change as age increases for the annual rate methods. The table below shows the relative gradients by exposure method that are required to produce no errors for a range of mortality rates.

Method	0.1%	1%	10%	50%
Traditional (BH)	-0.1%	-1%	-10%	-50%
Constant Force (CF)	0%	0%	0%	0%
Distribution (UD)	0.1%	1%	10%	50%

However, in a study, the mortality distribution for a given age, and hence the relative gradient, is determined by the mortality rate and the slope of the mortality curve at that age. If the relative gradient at an age happens to satisfy the conditions for a given exposure method, shown immediately after formula (4) above, then that method will not produce an error at that age. §6.2 looks at mortality distributions for sample ages that arise in a study and how these compare to the distributions implicit in the exposure methods.

More specifically, the relationship between the relative gradient and the rate determines which exposure method produces the smallest errors:

- If $\Delta_x < -½q_x$, use the annual rate method with traditional exposure,
- If $|\Delta_x| < +½q_x$, use the annual force method, and
- If $\Delta_x > +½q_x$, use the annual rate method with distributed exposure.

As the rates and relative gradients change over the range of ages in a study, no single exposure method is optimal for all ages.

For small rates, the relative gradient term dominates and the annual rate methods produce errors similar to the annual force method. When q_x is small, the generalized error can be approximated as:

$$e^G_{x+s,f} \approx T_{s,f} \Delta_x q_x \tag{5}$$

For large rates with large relative gradients, the three exposure methods will produce dramatically different errors. For large, negative relative gradients, the traditional exposure method will produce the smallest errors. For larger positive relative gradients, the distributed exposure method will generally produce the smallest errors. With the distributed exposure method, as rates become larger than the relative gradient, the error will change sign.

Later in this section, §6.3 Sample Table Errors will illustrate how the size of the errors vary by exposure method and how they change over a range of ages from a sample mortality table.

For an annual study using fractional exposure, the scale of the error for a year of age is determined not only by its rate and relative gradient, but also by its time factor, $T_{s,f}$. This time factor equals the time from the middle of the rate year to the middle of the fractional year of age; it affects both the size of the error and its sign.

In an annual study with exposure calculated for fractional years of age, the time from the middle of the rate year to the middle of the fractional period will be negative in the first half of the rate year and positive in the second half of the rate year. The sum of a rate year's time factors over all fractional periods will equal zero. That is, for P fractional periods of length f:

$$\Sigma_1^P T_{s,f} = \Sigma_1^P \big(s - ½(1-f)\big) = 0 \tag{6}$$

For an annual study with monthly exposure, the time from the middle of the rate year to the middle of each month is shown in the following chart:

$Month$	s	$T_{s,f}$
1	0.000	-0.4583
2	0.083	-0.3750
3	0.167	-0.2917
4	0.250	-0.2083
5	0.333	-0.1250
6	0.417	-0.0417
7	0.500	0.0417
8	0.583	0.1250
9	0.667	0.2083
10	0.750	0.2917
11	0.833	0.3750
12	0.917	0.4583

Substituting the generalized error formula (1) into the formula for exposure-weighted errors, §3.3 (15), which showed that the exposure-weighted sum of partial age errors over the full year is equal to zero, we have:

$$\Sigma_1^P \alpha_{x+s,f}\varepsilon_{x+s,f} \approx \Sigma_1^P \alpha_{x+s,f}T_{s,f}(\Delta_x + Mq_x)q_x \approx 0 \quad (7)$$

The generalized error formula is an estimate which ignores higher terms, resulting in symmetrical errors around the middle of the year of age. Even though the exposure-weights will reflect the true non-linearity of the errors, the equality from §3.3 (15) has been changed to an approximation.

Simplifying (7) shows that the exposure-weighted time factors are also approximately equal to zero, i.e.

$$\Sigma_1^P \alpha_{x+s,f}T_{s,f} \approx 0 \quad (8)$$

For the two partial ages arising from a full year of age being split by the calendar year-end, the two exposure-weighted errors will have equal absolute values and opposite signs. However, the rate errors for the two partial ages will not be equal in absolute value.

$$\varepsilon_{x,t} \approx T_{0,t}(\Delta_x + Mq_x)q_x \quad (9)$$

$$\varepsilon_{x+t,1-t} \approx T_{t,1-t}(\Delta_x + Mq_x)q_x \quad (10)$$

For each age anniversary date, the lives in the study will have partial ages of different lengths. To illustrate, assuming policies are issued on the first of the month and the age changes on the policy anniversary, the following table shows the time from the middle of the rate year to the middle of the partial year of age for twelve different policy anniversaries. For policy anniversaries falling on January 1st, the rate year falls completely within the study year, so there are no partial ages or associated errors to calculate. The table shows the exposure for each partial age, which is assumed to be uniformly distributed by month, to illustrate that the exposure weighted sum of the time factors is equal to zero.

Issue Date	t	$T_{0,t}$	$T_{t,1-t}$	$E_{x,t}$	$E_{x+t,1-t}$	$\Sigma(E_{x+s,f}T_{x+s,f})$
1st Jan	0.000					
1st Feb	0.083	-0.46	0.04	0.083	0.917	0.00
1st Mar	0.167	-0.42	0.08	0.167	0.833	0.00
1st Apr	0.250	-0.38	0.13	0.250	0.750	0.00
1st May	0.333	-0.33	0.17	0.333	0.667	0.00
1st Jun	0.417	-0.29	0.21	0.417	0.583	0.00
1st Jul	0.500	-0.25	0.25	0.500	0.500	0.00
1st Aug	0.583	-0.21	0.29	0.583	0.417	0.00
1st Sep	0.667	-0.17	0.33	0.667	0.333	0.00
1st Oct	0.750	-0.13	0.38	0.750	0.250	0.00
1st Nov	0.833	-0.08	0.42	0.833	0.167	0.00
1st Dec	0.917	-0.04	0.46	0.917	0.083	0.00

To illustrate the calculations using the error formula, the rates, relative gradients and resulting errors, results for sample ages from MNS ANB 2015 VBT are shown below. The calculations assume that policy anniversaries occur at mid-year, on average, which translates to:

$T_{0,½} = -¼$, and

$T_{½,½} = +¼$.

Using (9) and (10), the errors for two half-year partial ages can be estimated as:

$$\varepsilon_{x,½} \approx -¼(\Delta_x + Mq_x)q_x \quad (11)$$

$$\varepsilon_{x+½,½} \approx +¼(\Delta_x + Mq_x)q_x \quad (12)$$

The half-year errors shown below are based on attained age 70, with q_{70} = 1.147% and Δ_{70} = 11.2%. The values shown are time (i.e., $T_{0,½}$ or $T_{½,½}$), and the rate errors for the traditional, daily and distributed exposure methods.

Half-Year	Time	Traditional	Daily	Distributed
1	-0.25	-0.035%	-0.032%	-0.029%
2	+0.25	+0.035%	+0.032%	+0.029%

The next set of half-year errors show the results for attained age 90, with q_{90} = 13.69% and Δ_{90} = 12.2%:

Half-Year	Time	Traditional	Daily	Distributed
1	-0.25	-0.89%	-0.42%	+0.05%
2	+0.25	+0.89%	+0.42%	-0.05%

The final set of half-year errors are for issue age 70, policy year 1, with $q_{([70],1)}$ = 0.25% and $\Delta_{([70],1)}$ = 61%:

Half Year	Time	Traditional	Daily	Distributed
1	-0.25	-0.0384%	-0.0383%	-0.0381%
2	+0.25	+0.0384%	+0.0383%	+0.0381%

§6.2 Sample Age Distributions

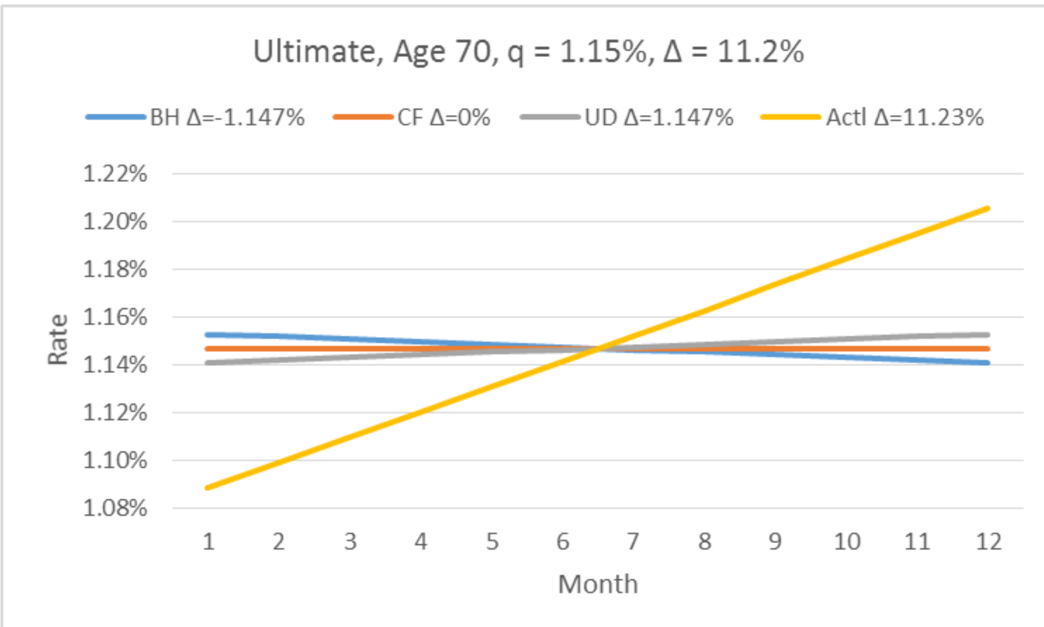
The following charts illustrate annualized monthly rates based on the distribution of deaths implicit in three exposure methods:

- Annual rate method with traditional exposure (BH for Balducci Hypothesis),
- Annual force method (CF for Constant Force), and
- Annual rate method with distributed exposure (UD for Uniform Distribution of Deaths).

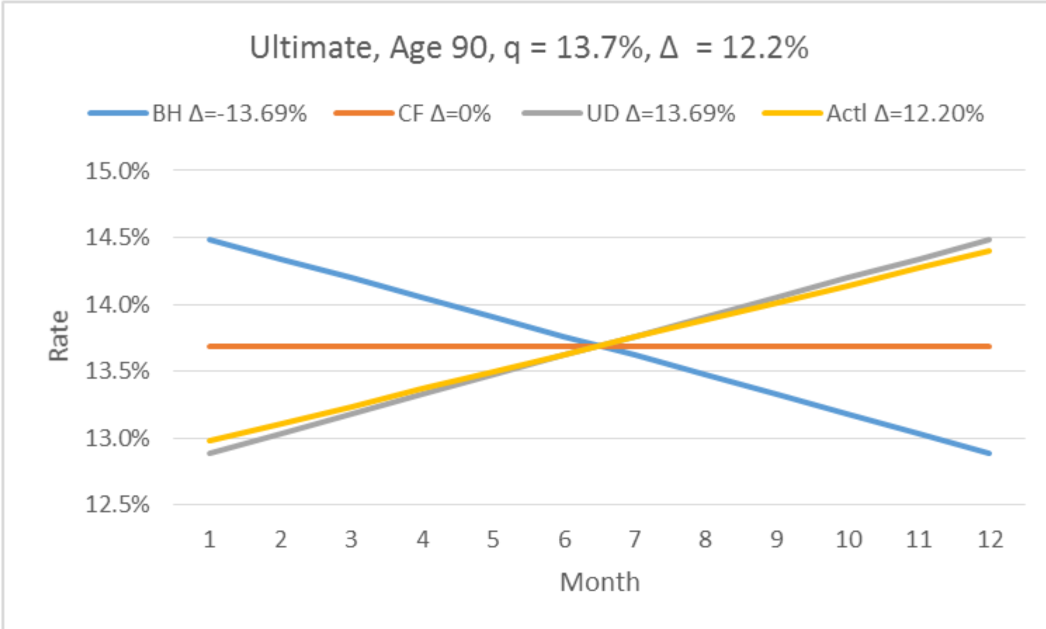
A fourth set of actual annualized monthly rates are also shown in in each chart. These "actual" rates were calculated using a distribution that reflects the actual relative gradient and rate at each rate.

Three ages are illustrated, using sample rates from MNS ANB 2015 VBT.

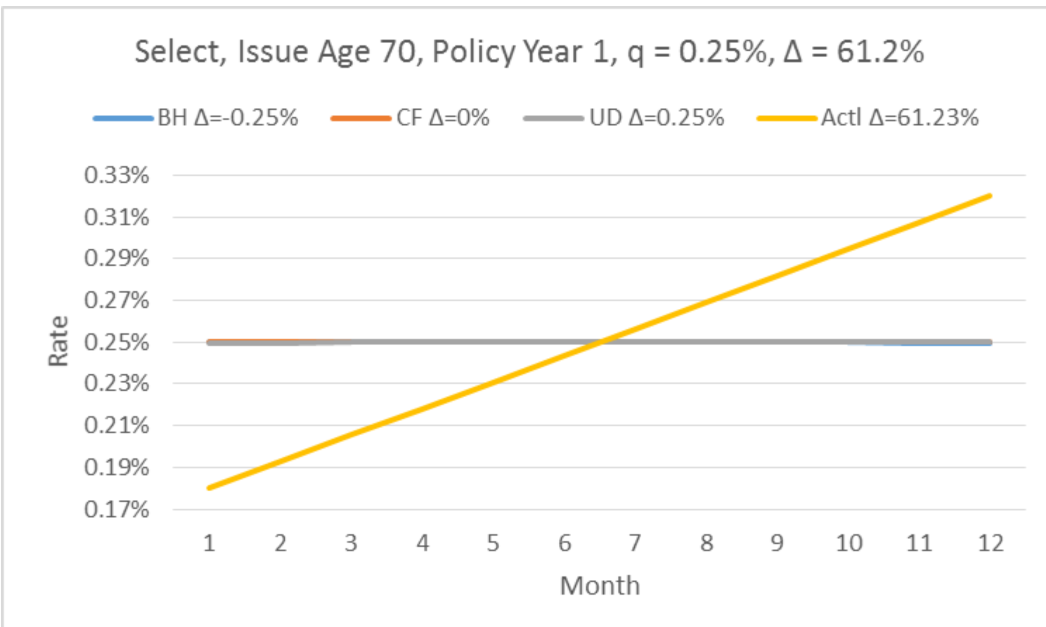
At age 70, while there are discernible differences between the annualized rates for the three exposure methods, there is a much larger difference when compared to actual rates. Rates for the uniform distribution of deaths are closest to actual, but are not close.



At age 90, the rate has increased to 14% and is slightly larger than its relative gradient. This relationship causes the actual rates to be very close to the rates associated with a uniform distribution of deaths. Meanwhile, the relative gradient for the Balducci hypothesis, at -14%, is causing its rates to decline with increasing monthly age.

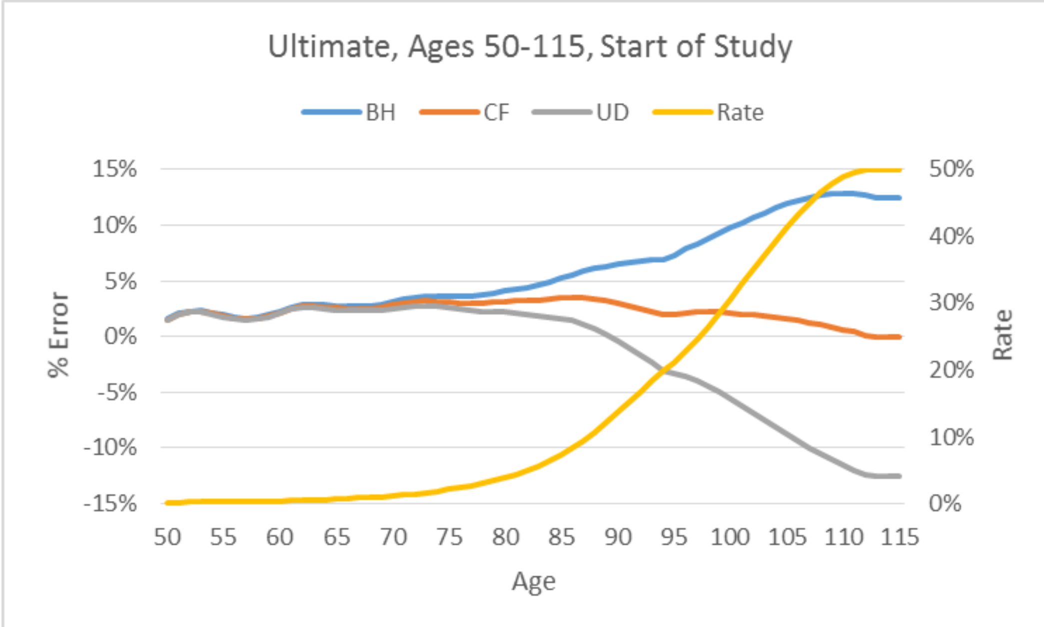


For issue age 70 in the first policy year, the actual rates are considerably steeper than the rates for all three exposure methods, which are indistinguishably flat in the graph below.



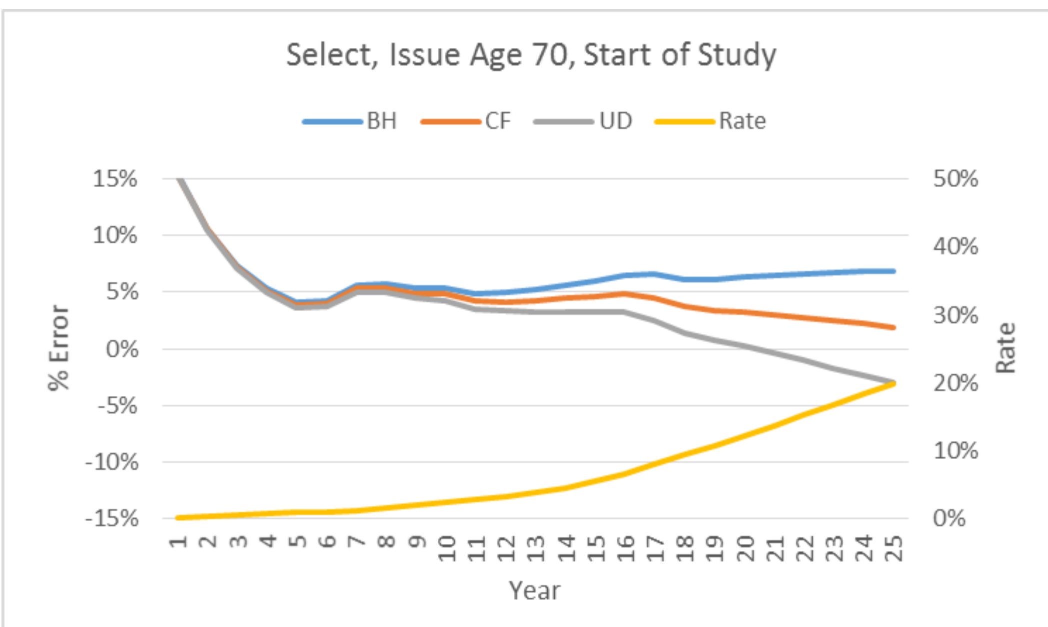
§6.3 Sample Table Errors

In the following graph, the percentage errors for the partial ages at the start of the study period are shown for all three exposure methods, using ultimate ages 50 to 115 from MNS ANB 2015 VBT. The "% Error" scale on the left-hand side of the graph applies to the BH, CF and UD exposure methods while the "Rate" scale on the right-hand side applies to the rate itself, which ranges from less than 0.01 to 0.50.



The above graph shows that the percentage errors are similar below age 70, ranging from 2% to 3%, for all three methods. Above age 70, the percentage errors for the traditional exposure method increase up to 12% at age 112 when the mortality rate reaches 50%. For the constant force method, the percentage error decreases from 3% at age 70 to 0% at age 112. For the distributed exposure method, the percentage error reduces to 0% at age 89 and then becomes increasing negative, reaching -12% at age 112. This method has the smallest error of the three methods up to age 93, after which the constant force method has the smallest error.

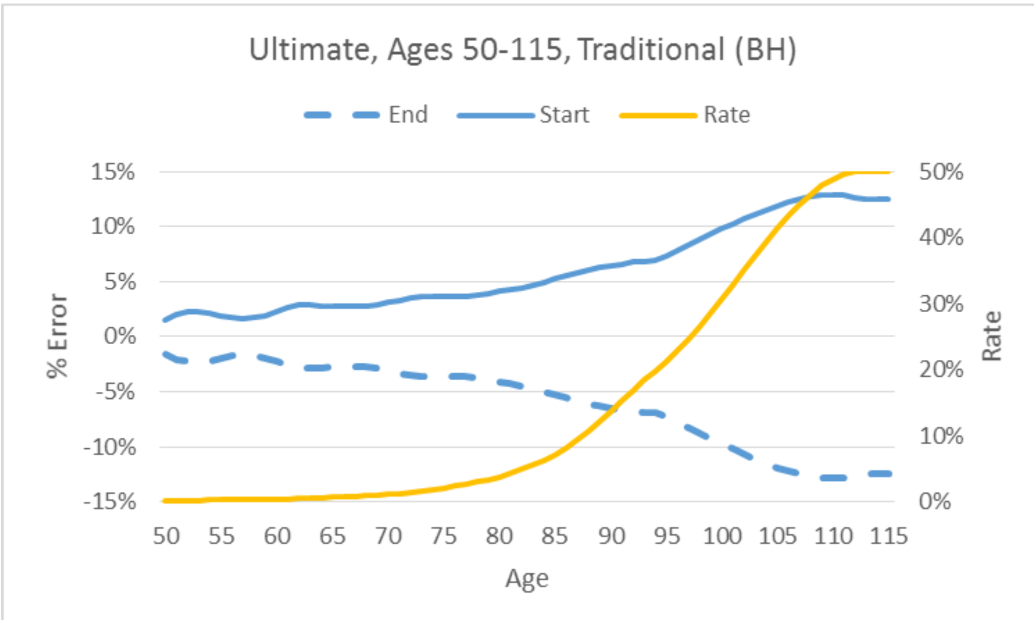
The next graph uses the same format as the preceding graph but illustrates select rates for issue age 70 for policy years 1 through 25. While the rates are lower, the errors show a pattern similar to that shown in the preceding graph.

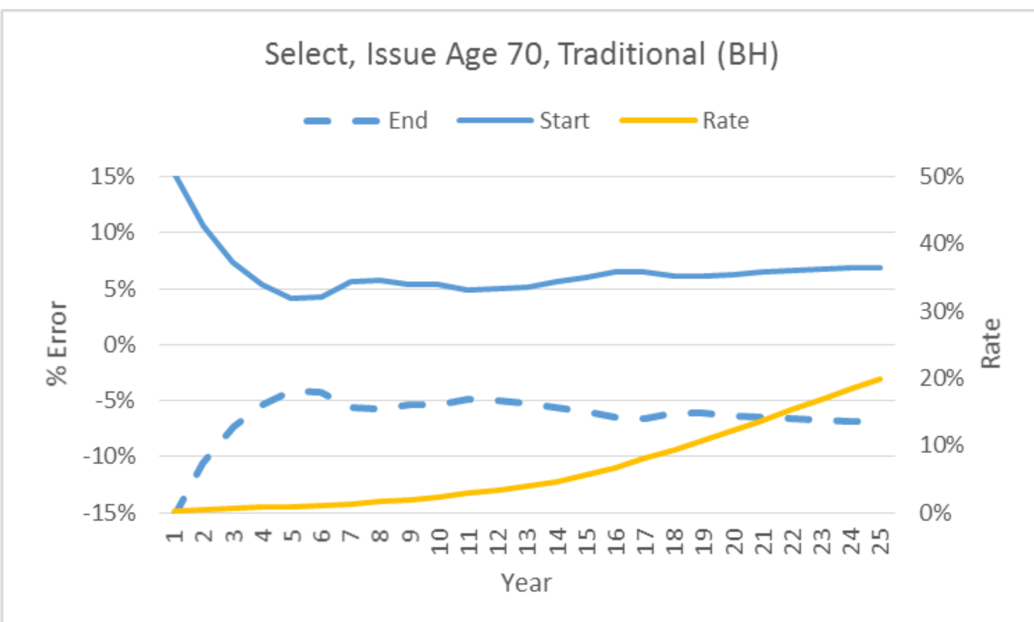


For select rates, the percentage error varies considerably by issue age, particularly in the first policy year, as shown in the table below for sample issue ages. These errors percentages are much higher than those for ultimate rates due to the rapid increase in mortality rates during the first few policy years after issue.

Issue Age	Policy Year 1 % Error
[50]	10.5%
[70]	15.4%
[90]	31.0%

The percentage errors for the partial ages at the start and end of the study are shown below for the traditional exposure method only. Notice how the curves for the start and end of the study are mirror images. The first graph shows ultimate rates while the next graph shows select rates.

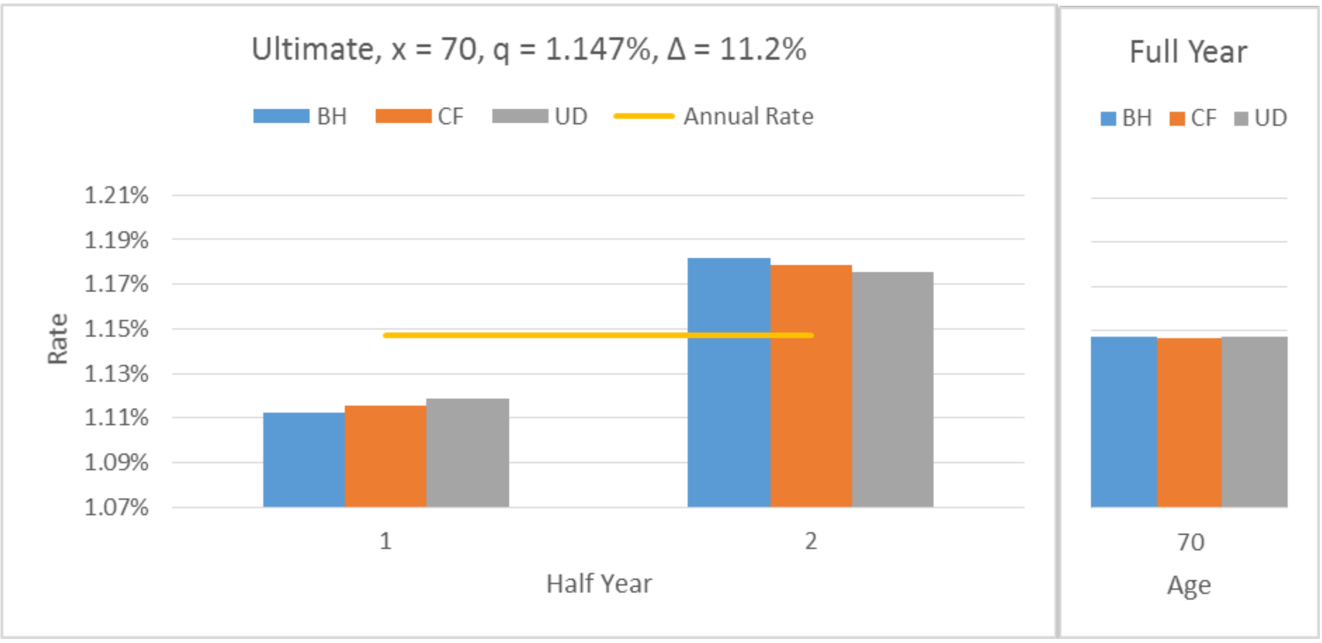




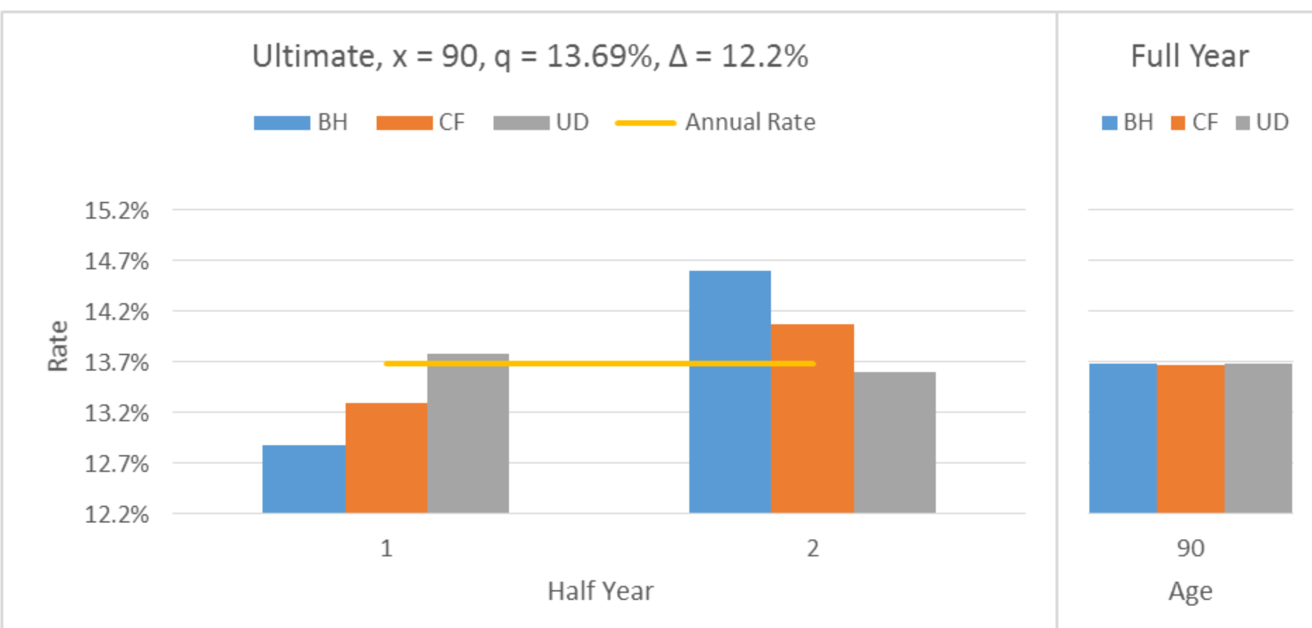
§6.4 Projected Sample Age Errors

Annual rates for half-years of age are illustrated in the following three graphs. The yellow line indicates the true annual rate while the three groups of bars show rates for the first half-year, second half-year and full year for each of the three exposure methods.

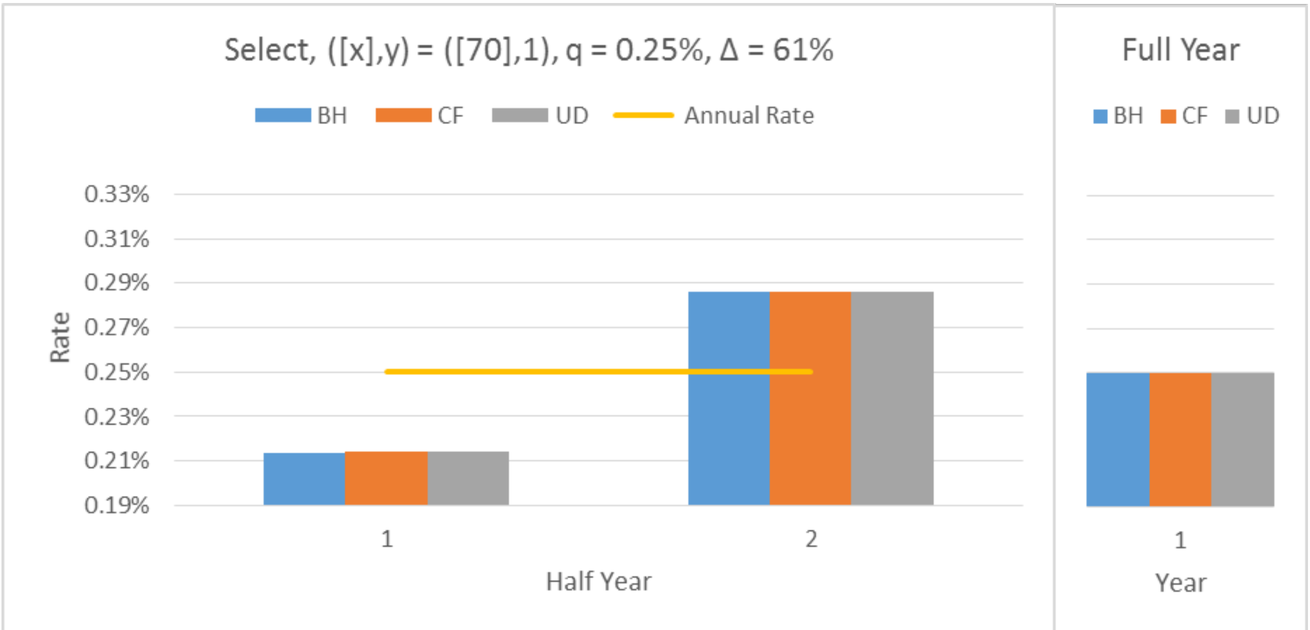
The first graph illustrates ultimate age 70. As expected, the three study methods produce similar errors for each half-year, with the distributed method the closest to and the traditional method the farthest from the true annual rate. Note that the relative gradient is almost 10 times the rate.



The next graph shows ultimate age 90, where there is a more significant spread in the errors. Again, the distributed method is the closest, the traditional method is the farthest, but the errors are now opposite in sign, since the rate now exceeds the relative gradient.

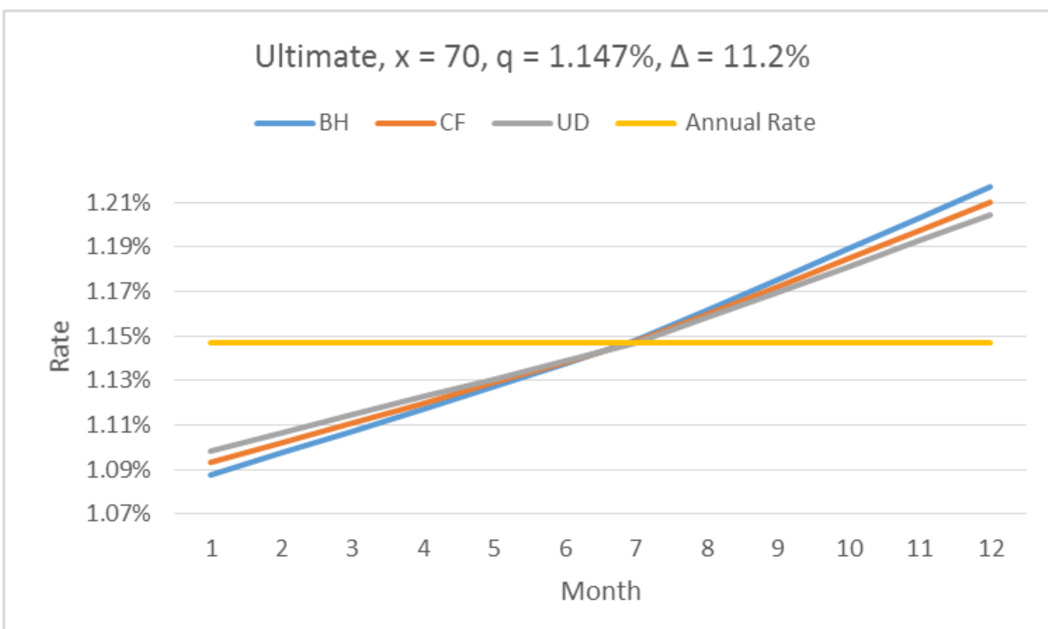


The last of the three graphs illustrates issue age 70, policy year 1. The annual rates by half-year produced by all three methods are indistinguishable and not close to the actual rate, because the relative gradient is more than 200 times the rate.

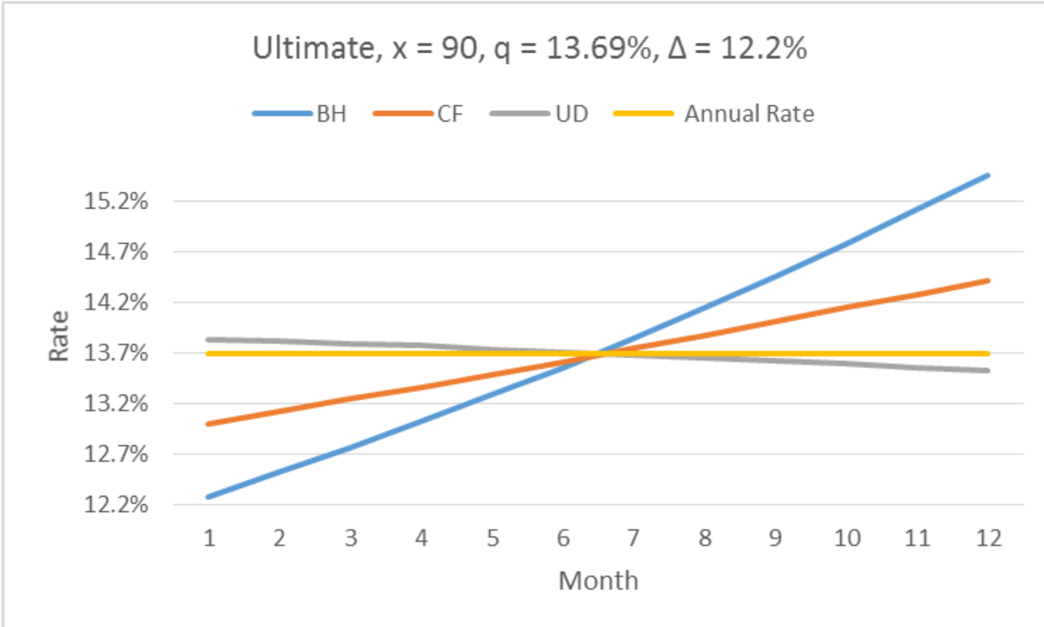


The next three graphs illustrate annual rates by month for the same three sample ages shown above. The yellow line shows the true annual rate. The annual rates for the three exposure methods use the bi-linear force distribution, which is why the lines connecting the annual rates change slope at an inflexion point around month 7, where annual rates cross the true annual rate.

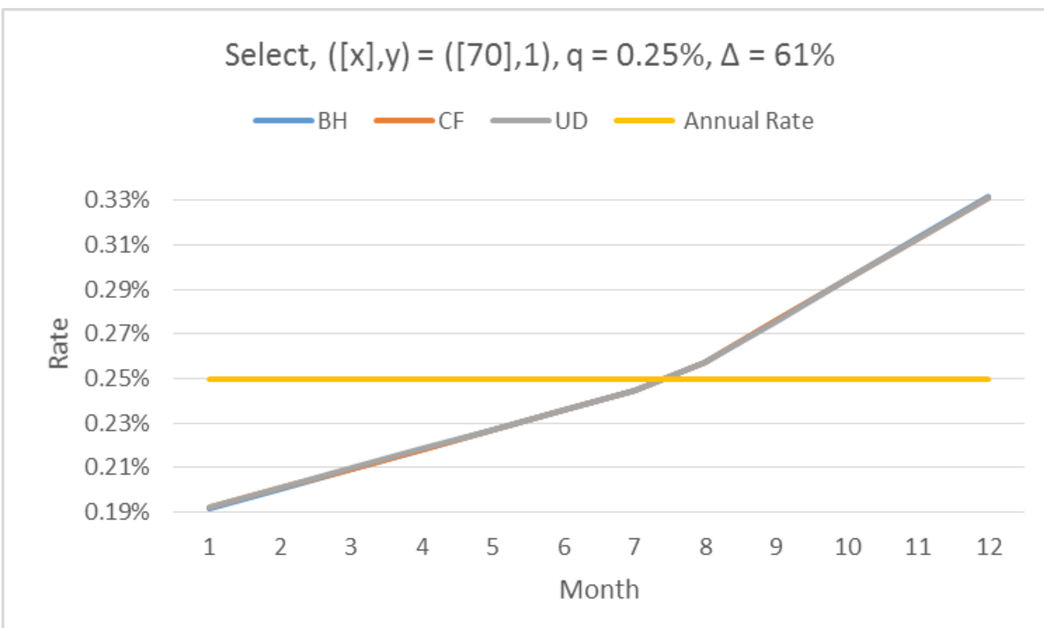
For ultimate age 70, the three exposure methods produce fairly similar annual rates.



For ultimate age 90, the three exposure methods produce dramatically different patterns of annual monthly rates: Distributed exposure shows rates that slightly decline, traditional exposure shows rates that increase by about 2% per month and constant force rates are halfway between the two.



For issue age 70, policy year 1, the three exposure methods produce rates that are so close that only the UD line is visible.



§6.5 Hybrid Annual Rate Method

The hybrid exposure method, introduced in Chapter 3, uses traditional exposure for partial ages at the start of the study and distributed exposure for the partial ages at the end of the study. The following table shows the errors at the start and end of an N-year study for the traditional, distributed and hybrid methods, assuming uniformly distributed anniversaries. The errors originate from the first and last cohorts which contribute exposure to the age, with notation defined in §4.1.

Error	Traditional	Distributed	Hybrid
Study Period Start: $\varepsilon_{x+½,½,1}$	$+¼(\Delta_x + q_x)q_x$	$+¼(\Delta_x - q_x)q_x$	$+¼(\Delta_x + q_x)q_x$
Study Period End: $\varepsilon_{x,½,\mathrm{N}+1}$	$-¼(\Delta_x + q_x)q_x$	$-¼(\Delta_x - q_x)q_x$	$-¼(\Delta_x - q_x)q_x$

For the traditional and distributed methods, the two errors are equal and opposite in sign. If the first and last cohorts are equal in size, these errors will cancel out to give a zero net error. For the hybrid method, the errors will have the same sign. If the first and last cohorts are equal in size, the total error will be double.

The error in the rate for age *x* using the hybrid method and assuming equal cohorts is:

$$\varepsilon_x = ½\left(\varepsilon_{x+½,½,1} + \varepsilon_{x,½,\mathrm{N}+1}\right)/N \quad (1)$$

Substituting the hybrid error formula gives:

$$\varepsilon_x = ½\,(¼(\Delta_x + q_x)q_x + ¼(\Delta_x - q_x)q_x)/N \quad (2)$$

Simplifying (2) shows the error is proportional to the square of the rate and inversely proportional to the number of study years.

$$\varepsilon_x = ¼q_x^2/N \quad (3)$$

So, given cohorts equal in size, the traditional and hybrid methods will result in no error, while the hybrid method introduces errors that are insignificant for sufficiently small values of q_x but potentially quite significant for large values of q_x.

Using ultimate rates from MNS ALB 2015 VBT, the following table shows that errors for the hybrid exposure method for a three-year study. Errors are not material for ages 50 and 70, but for a three-year study, by age 90 the error is 1%, and increases to over 4% at age 112 where the mortality rate reaches 50%.

$\boldsymbol{x}$	50	70	90	112
$\boldsymbol{q_x}$	0.192%	1.147%	13.690%	50.000%
$\boldsymbol{\varepsilon/q}$	0.016%	0.096%	1.141%	4.167%

§6.6 Increasing Cohorts

In §4.4 the, following formula was developed for the errors accumulating in an N-year study with cohort size increasing simply at an annual rate, i:

$$\varepsilon_x = -\,\alpha_{x+t,1-t}\varepsilon_{x+t,1-t}Ni/\left(N + ½(N+1)Ni - \alpha_{x+t,1-t}Ni\right) \quad (1)$$

A three-year study using the annual rate method with traditional exposure will be used to illustrate the range of errors arising due to increasing cohorts, assuming birthdays are uniformly distributed through the calendar year and the distribution of deaths is the same for all cohorts. Applying these assumptions to §6.1 (3), we have:

$$\varepsilon_{x+½,½} = -\varepsilon_{x,½} = ¼(\Delta_x + q_x)q_x\,, \quad (2)$$

and

$$\alpha_{x+½,½} = 1 - \alpha_{x,½} \,. \qquad (3)$$

The following table shows the overall ultimate mortality rate, relative gradient, error, percentage error, ultimate mortality rates for the first and last cohorts, and the exposure weight for the first half-year of age at the start of the study for ultimate ages 50, 70, 90, and 112.

x	50	70	90	112
q_x	0.192%	1.147%	13.690%	50.000%
Δ_x	6.0%	11.2%	12.2%	0.0%
$\varepsilon_{x+½,½}$	0.003%	0.04%	0.89%	6.25%
ε/q	2%	3%	6%	13%
$q_{x+½,½,1}$	0.195%	1.18%	14.58%	56.25%
$q_{x,½,N+1}$	0.189%	1.11%	12.80%	43.75%
$\alpha_{x+½,½}$	49.95%	49.76%	46.71%	35.35%

The following table shows percentage errors accumulating in the study based on the ultimate rates from the preceding table combined with six different cohort growth rates.

Increase		% Study Error			
Annual	3 Yr Study	50	70	90	113
0%	0%	0.000%	0.000%	0.000%	0.000%
1%	3%	-0.008%	-0.015%	-0.029%	-0.043%
5%	15%	-0.036%	-0.072%	-0.139%	-0.204%
10%	30%	-0.067%	-0.134%	-0.259%	-0.379%
50%	150%	-0.221%	-0.439%	-0.845%	-1.212%
100%	300%	-0.309%	-0.615%	-1.178%	-1.670%

The following table shows the select rate, relative gradient and error together with select rates for the first and last cohorts, and the exposure weight for issue ages 50, 70 and 90 and select year 1.

$[x], y$	[50],1	[70],1	[90],1
$q_{[x],1}$	0.052%	0.250%	2.069%
$\Delta_{[x],1}$	41.9%	61.2%	125.0%
$\varepsilon_{[x],1+½,½,}$	0.005%	0.04%	0.66%
ε/q	10%	15%	32%
$q_{[x],1+½,½,1}$	0.057%	0.29%	2.73%
$q_{[x],1,½,N+1}$	0.047%	0.21%	1.41%
$\alpha_{[x],1+½,½}$	49.98%	49.96%	49.97%

The following table shows percentage errors accumulating in the study based on the select rates from the preceding table combined with six different growth rates. These percentage errors for select rates are much higher than those for ultimate rates due to the rapid increase in mortality rates during the first few years after issue.

Increase		% Study Error		
Annual	3 Yr Study	[50],1	[70],1	[90],1
0%	0%	0.000%	0.000%	0.000%
1%	3%	-0.052%	-0.076%	-0.156%
5%	15%	-0.244%	-0.357%	-0.738%
10%	30%	-0.456%	-0.668%	-1.380%
50%	150%	-1.497%	-2.194%	-4.534%
100%	300%	-2.096%	-3.071%	-6.347%

In summary, for studies where cohorts are not equal in size for a given age, there may be material errors in some or all ages. As was shown, these errors can be quantified and, if material, then either:

- The partial ages can be excluded from the study by using a rate-year study period, or
- The rates can be adjusted to correct for the bias arising from the study method.

§6.7 Error Formula Derivation

The error in the annual force for a partial age using the annual force method was given in §3.4 (11) as:

$$\xi_{x+s,f} = \bar{\mu}_{x+s,f} - \bar{\mu}_x \, . \tag{1}$$

Using §5.3 (8) to substitute for $\bar{\mu}_{x+s,f}$ (using the linear force distribution) into the above formula results in the following:

$$\xi_{x+s,f} = \bar{\mu}_x + T_{s,f}\Delta\mu_x - \bar{\mu}_x = T_{s,f}\Delta\mu_x = T_{s,f}\Delta_x\bar{\mu}_x \tag{2}$$

The above formula shows that the error can be expressed as the product of $T_{s,f}$, the relative gradient and the annual force.

In the first part of the following equation, the formula for the error in the annual rate for a partial age, from §3.3 (13), is shown with a superscript of CF added to denote constant force. In the next part, §3.4 (2) is used to substitute for $q_{x+s,f}$. In the final part of the equation below, $1 - q_x$ is substituted for $e^{-\bar{\mu}_x}$, using §2.1 (1), and terms are regrouped.

$$\varepsilon^{CF}_{x+s,f} = q^{CF}_{x+s,f} - q_x = 1 - e^{-(\bar{\mu}_x + T_{s,f}\Delta\mu_x)} - q_x = (1 - q_x)\left(1 - e^{-T_{s,f}\Delta\mu_x}\right) \tag{3}$$

Expanding the exponential term, ignoring higher order terms and rearranging gives:

$$\varepsilon^{CF}_{x+s,f} \approx -T_{s,f}\Delta\mu_x(1 - q_x) \tag{4}$$

The increase in force, $\Delta\mu_x = \Delta_x\bar{\mu}_x$, can be re-expressed in terms of the mortality rate and the relative gradient by using §2.1 (2) to substitute $-\log_e(1 - q_x)$ for $\bar{\mu}_x$:

$$\Delta\mu_x = -\Delta_x \log_e(1 - q_x) \tag{5}$$

Using the Taylor expansion, i.e., $\log_e(1 - X) = -X - X^2/2 - X^3/3 - \cdots$, to eliminate the natural logarithm, the increase in force can be estimated by ignoring second and higher powers as:

$$\Delta\mu_x \approx -\Delta_x q_x \tag{7}$$

Substituting (7) into (4) gives:

$$\varepsilon^{CF}_{x+s,f} \approx T_{s,f}\Delta_x q_x(1 - q_x) \tag{8}$$

Ignoring the second power of q_x, i.e. substituting $1 - q_x \approx 1$,the error is given by:

$$\varepsilon_{x+s,f}^{CF} \approx T_{s,f}\Delta\mu_x q_x \tag{9}$$

The relationship between the annual rate calculated in the study, $q_{x+s,f}^{CF}$, and the true underlying annual rate, q_x, is:

$$q_{x+s,f}^{CF} \approx \left(1 + T_{s,f}\Delta\mu_x\right)q_x \tag{10}$$

The error in the annual rate for a partial age using the traditional exposure method, $\varepsilon_{x+s,f}^{BH} = q_{x+s,f}^{BH} - q_x$, can alternatively be expressed as the difference between $q_{x+s,f}^{BH}$ and $q_{x+s,f}^{CF}$ plus the error for the constant force method, $\varepsilon_{x+s,f}^{CF}$, as follows:

$$\varepsilon_{x+s,f}^{BH} = q_{x+s,f}^{BH} - q_x = q_{x+s,f}^{BH} - q_{x+s,f}^{CF} + \varepsilon_{x+s,f}^{CF}. \tag{11}$$

Substituting using the approximations shown in §3.3 (4) and §3.4 (16) to convert annual rates to fractional rates gives:

$$\begin{aligned}\varepsilon_{x+s,f}^{BH} &\approx \left({}_f q_{x+s}/f + \left(s - (1-f)\right)\left({}_f q_{x+s}/f\right)^2\right) \\ &\quad - \left({}_f q_{x+s}/f - ½(1-f)\left({}_f q_{x+s}/f\right)^2\right) + \varepsilon_{x+s,f}^{CF}\end{aligned} \tag{12}$$

Combining terms, (12) simplifies to:

$$\varepsilon_{x+s,f}^{BH} \approx T_{s,f}\left({}_f q_{x+s}/f\right)^2 + \varepsilon_{x+s,f}^{CF} \tag{13}$$

Substituting for ${}_f q_{x+s}$ in (13), using the first order estimate shown in §3.4 (18)), and then substituting for $\varepsilon_{x+s,f}^{CF}$, using (9) above, gives:

$$\varepsilon_{x+s,f}^{BH} \approx T_{s,f}(\Delta_x + q_x)q_x \tag{14}$$

Similar to (11), the error in the annual rate for a partial age using the distributed exposure method can be alternatively expressed as the difference between $q_{x+s,f}^{UD}$ and $q_{x+s,f}^{CF}$ plus the error for the constant force method, $\varepsilon_{x+s,f}^{CF}$, as follows:

$$\varepsilon_{x+s,f}^{UD} = q_{x+s,f}^{UD} - q_x = q_{x+s,f}^{UD} - q_{x+s,f}^{CF} + \varepsilon_{x+s,f}^{CF} \tag{15}$$

Substituting using the approximations shown in §3.3 (8) and §3.4 (16) to convert annual rates to fractional rates gives:

$$\begin{aligned}\varepsilon_{x+s,f}^{UD} &\approx \left({}_f q_{x+s}/f - s\left({}_f q_{x+s}/f\right)^2\right) \\ &\quad - \left({}_f q_{x+s}/f - ½(1-f)\left({}_f q_{x+s}/f\right)^2\right) + \varepsilon_{x+s,f}^{CF}\end{aligned} \tag{16}$$

Combining terms, (16) simplifies to:

$$\varepsilon_{x+s,f}^{UD} \approx -T_{s,f}\left({}_f q_{x+s}/f\right)^2 + \varepsilon_{x+s,f}^{CF} \tag{17}$$

Substituting for ${}_f q_{x+s}$ in (17), using the first order estimate shown in §3.4 (18)), and then substituting for $\varepsilon_{x+s,f}^{CF}$, using (9) above, gives:

$$\varepsilon_{x+s,f}^{UD} \approx T_{s,f}(\Delta_x - q_x)q_x \tag{18}$$

Equations (9), (14) and (18) can be merged by defining a method factor M such that

- $M = 0$ for the annual force method,
- $M = 1$ for the annual rate method with traditional exposure, and
- $M = -1$ for the annual rate method with distributed exposure.

The generalized error formula is shown below, with a superscript of G indicating a generalized approach that applies to multiple exposure methods.

$$\varepsilon_{x+s,f}^{G} \approx T_{s,f}(\Delta_x + Mq_x)q_x \tag{19}$$

Chapter 7: Annual Force with Weighted Exposure

The partial-year errors that have been examined were due to differences between actual and assumed mortality distributions. Using the annual force method, such errors can be eliminated by adjusting the exposure to reflect the actual distribution of deaths in the study.

Formula §3.4 (11) states that the annual force for a partial age, $\bar{\mu}_{x+s,f}^{CF}$, is equal to the actual annual force, $\bar{\mu}_x$, plus the error in the annual force for the partial age, $\xi_{x+s,f}$. Replacing the error using §6.7 (2), yields the final part of the equation below:

$$\bar{\mu}_{x+s,f}^{CF} = \bar{\mu}_x + \xi_{x+s,f} = \bar{\mu}_x\left(1 + T_{s,f}\Delta_x\right) \qquad (1)$$

The annual force for a partial age can also be expressed as deaths divided by exposure, as in §3.4 (1):

$$\bar{\mu}_{x+s,f}^{CF} = d_{x+s,f}/E_{x+\mathrm{s},f}^{CF} \qquad (2)$$

Equating (1) and (2) and solving for the annual force gives:

$$\bar{\mu}_x = d_{x+s,f}/\left(\left(1 + T_{s,f}\Delta_x\right)E_{x+\mathrm{s},f}^{CF}\right) \qquad (3)$$

We will define $E_{x+\mathrm{s},f}^{LF}$ as the exposure for force weighted by the linear force distribution, as follows:

$$E_{x+\mathrm{s},f}^{LF} = \left(1 + T_{s,f}\Delta_x\right)E_{x+\mathrm{s},f}^{CF} \qquad (4)$$

Substituting (4) into (3) gives:

$$\bar{\mu}_x = d_{x+s,f}/E_{x+\mathrm{s},f}^{LF} \qquad (5)$$

The above formula can be interpreted as follows: If the annual force for a partial age is calculated using exposure weighted for the linear force distribution, i.e., $E_{x+\mathrm{s},f}^{LF}$, then that annual force will be without error if the actual mortality distribution matches the linear force distribution. In other words, we can adjust exposure to minimize the error associated with partial ages.

While (5) indicates that deaths divided by weighted exposure is identically equal to the true annual force, it is just an estimate that only holds true if the linear force distribution with its assumed relative gradient is accurate.

Using the partial age notation in a similar fashion to §3.4, let $\bar{\mu}_{x+s,f}^{LF}$ be the annual force for a partial age with exposure weighted for the linear force distribution:

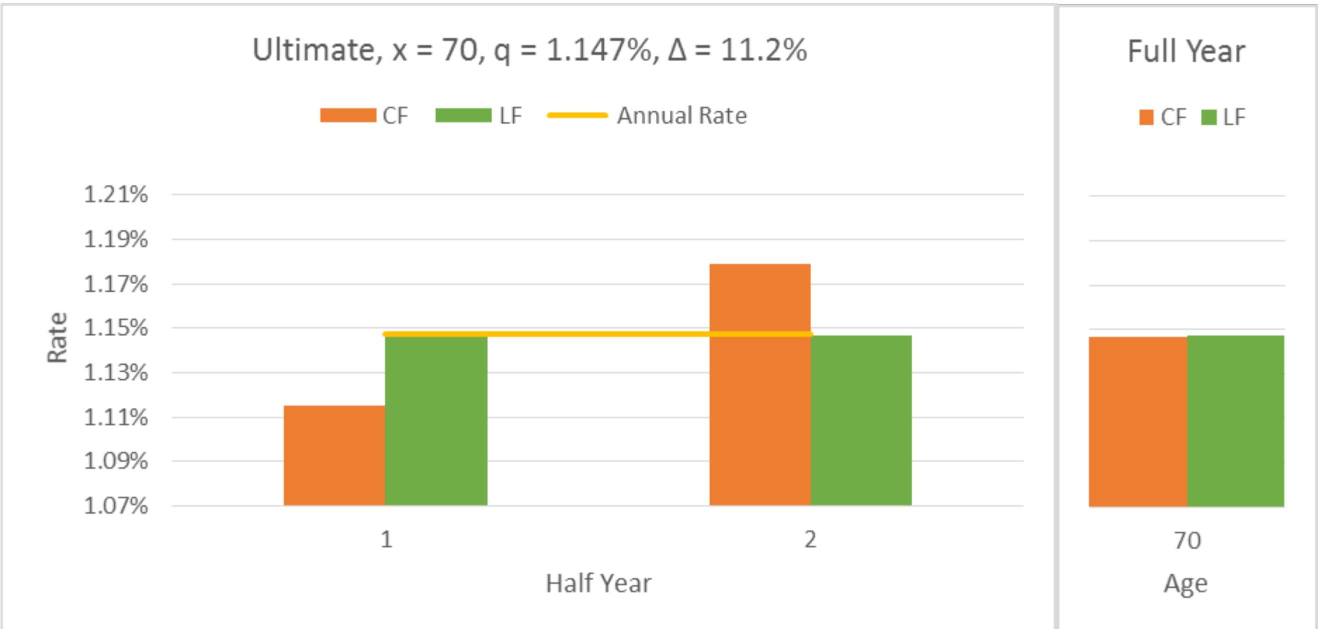
$$\bar{\mu}_{x+s,f}^{LF} = d_{x+s,f}/E_{x+\mathrm{s},f}^{LF} \qquad (6)$$

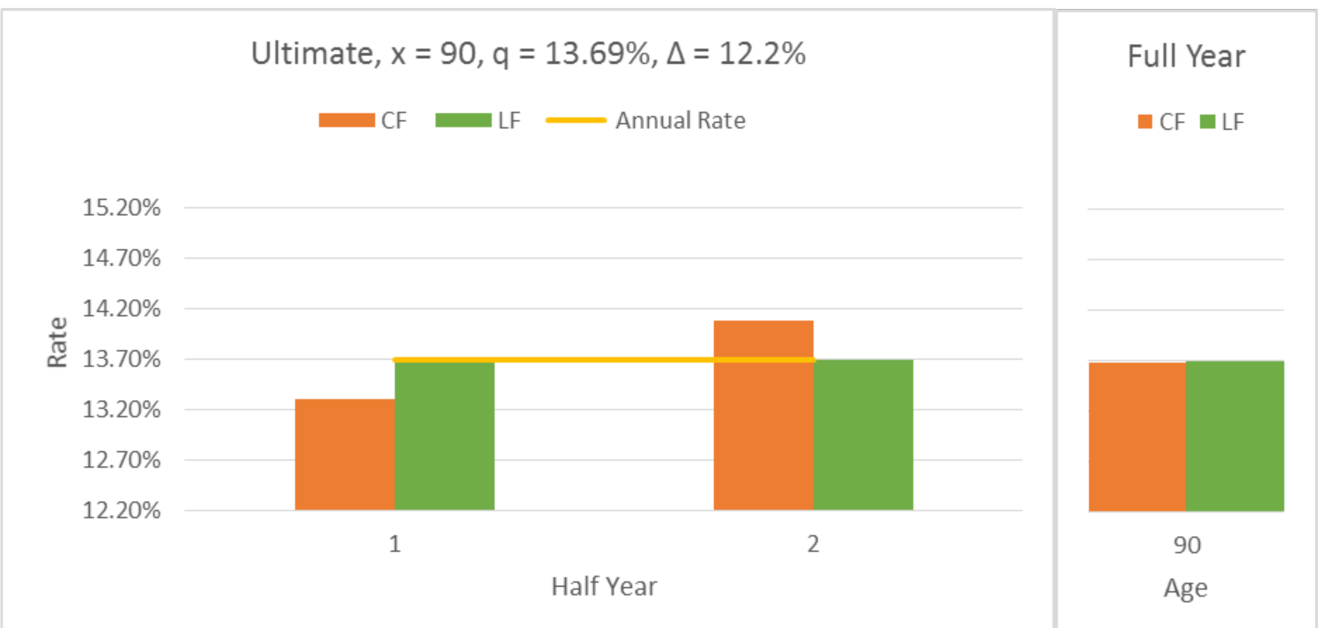
Then the annual force for a partial age with weighted exposure will equal the annual force for the full year of age without error if the linear force distribution is accurate:

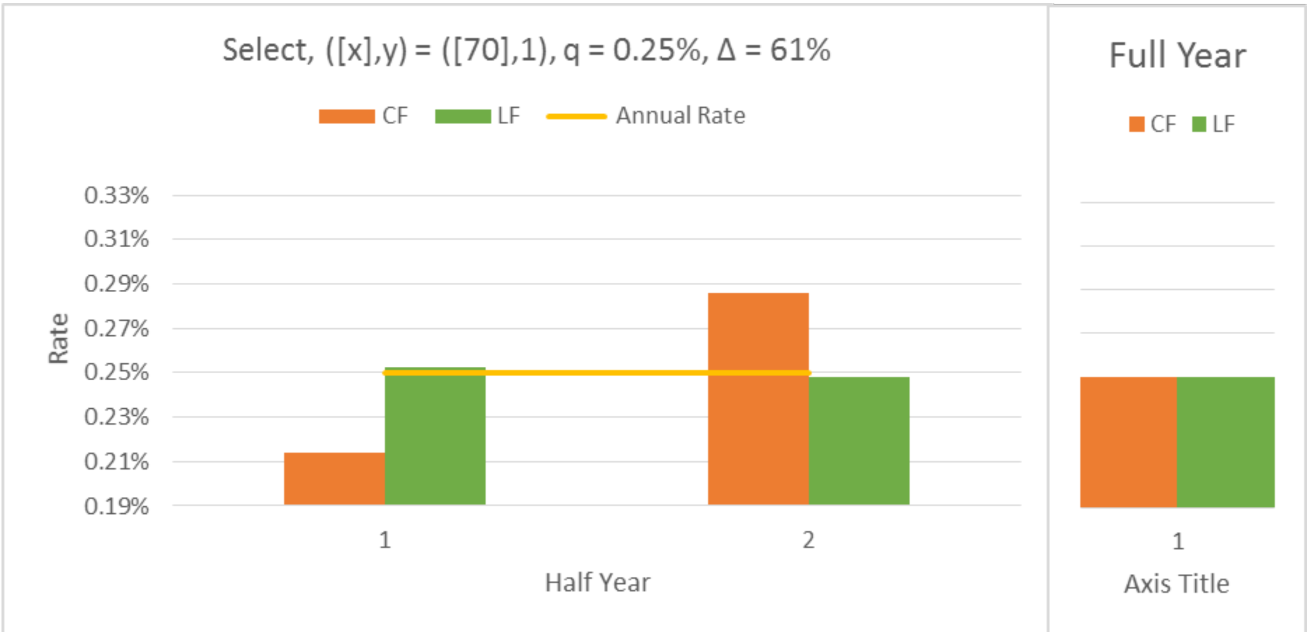
$$\bar{\mu}_{x+s,f}^{LF} = \bar{\mu}_x \qquad (7)$$

The following charts for the usual three ages show annual rates by half-year calculated with exposure weighted by the linear force distribution (designated by "LF") and without exposure weighted by the linear force distribution (designated by "CF"). The underlying distribution of lives and deaths is based on the more advanced bi-linear distribution, which was developed in §5.4 to be both consistent and continuous.

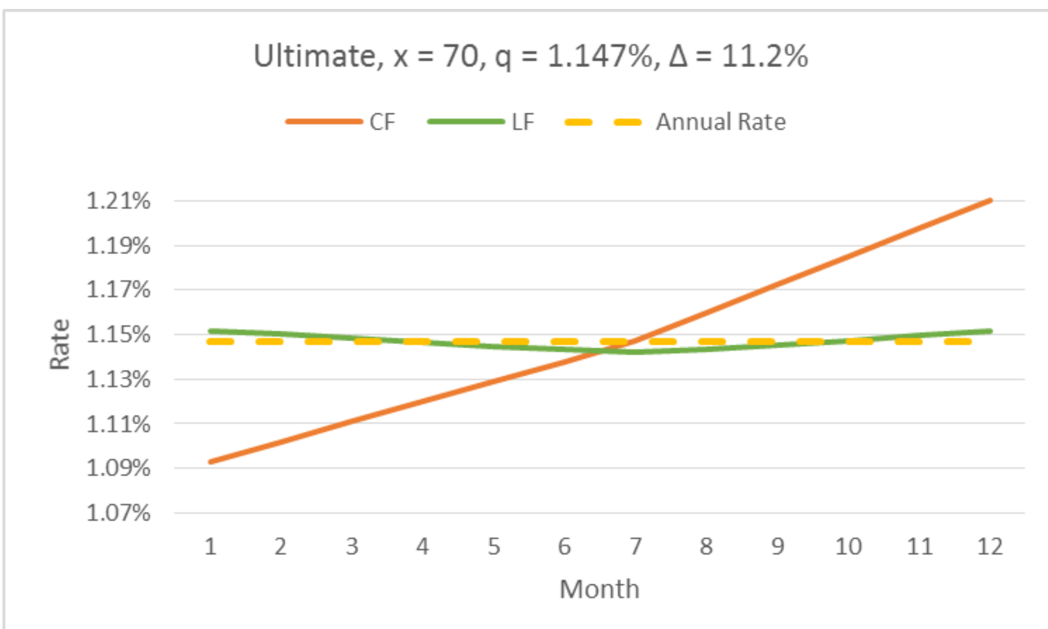
In each case, the CF rates show material errors in both half-years while both LF half-year rates closely reproduce the actual rate. In the third chart, for issue age 70, policy year 1, there is a small error in the LF annual rates due to the difference between the linear force distribution, assumed in the weighted exposure, and the bi-linear force distribution, used to project lives and deaths. To the far right of each chart, you will see that both methods reproduce the full-year rate.

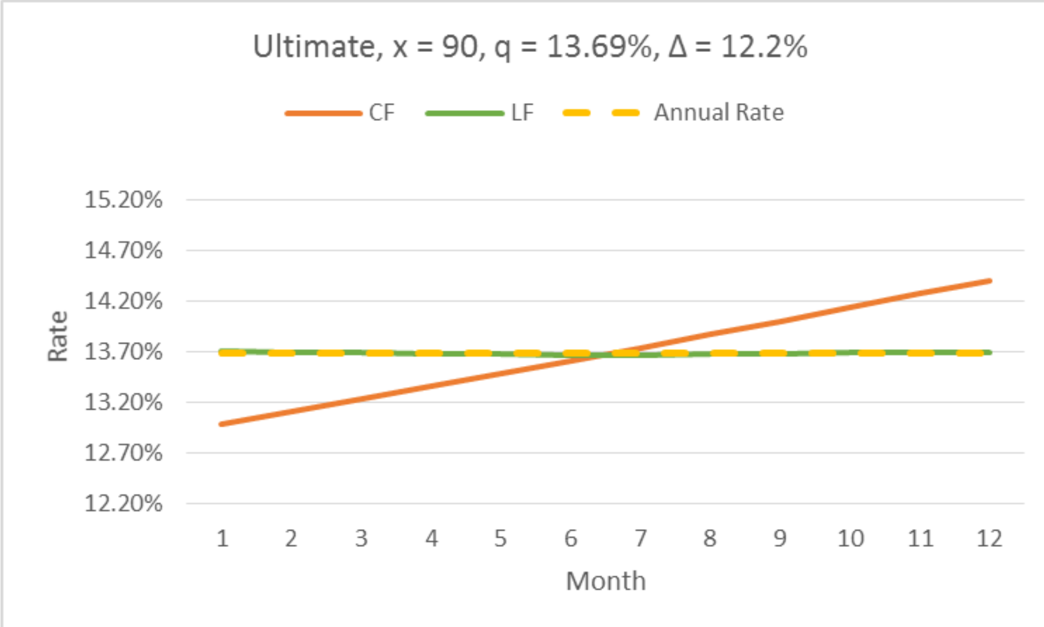




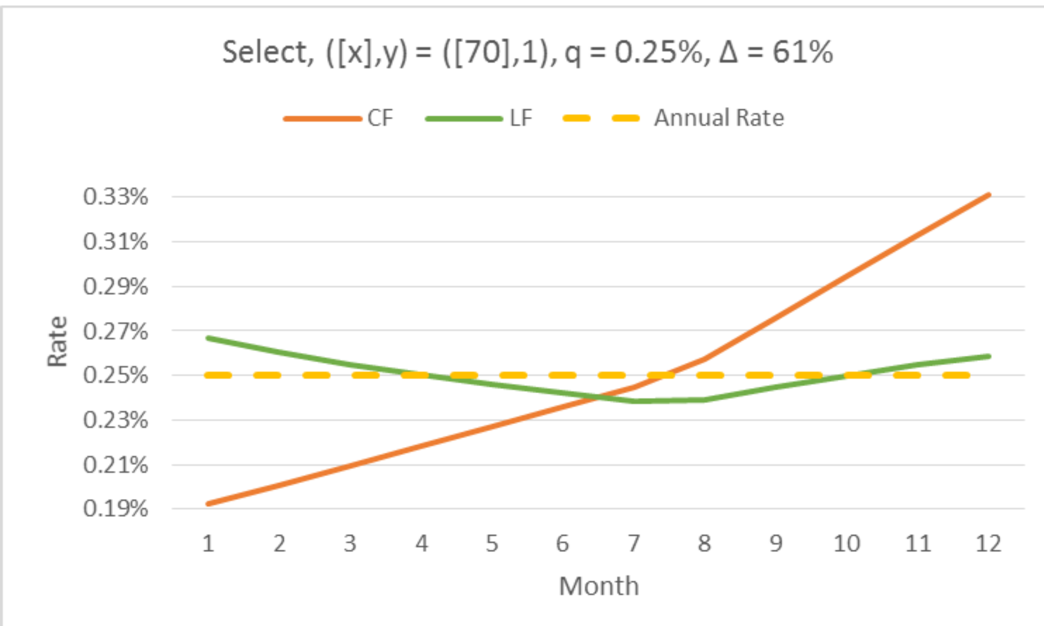


The following graphs illustrate annual CF (unweighted exposure) and LF (weighted exposure) rates by month for the same three ages. As the annual rates using weighted exposure closely match the annual rate in all months, a dashed line is used to distinguish the annual rate line. As noted previously, while the rates calculated using weighted exposure rates eliminate most of the errors shown by the CF rates, there is some residual error due to the difference between the linear force distribution, assumed in the weighted exposure, and the bi-linear force used to project lives and deaths.





Ultimate, x = 90, q = 13.69%, Δ = 12.2%
CF
LF
Annual Rate
15.20%
14.70%
14.20%
13.70%
13.20%
12.70%
12.20%
Rate
1
2
3
4
5
6
7
8
9
10
11
12
Month



Select, ([x],y) = ([70],1), q = 0.25%, Δ = 61%
CF
LF
Annual Rate
0.33%
0.31%
0.29%
0.27%
0.25%
0.23%
0.21%
0.19%
Rate
1
2
3
4
5
6
7
8
9
10
11
12
Month

Chapter 8: Glossary

§8.1 Terminology

Annual Rate Method: Calculates the mortality rate for an age as the number of deaths for the age divided by the number of lives in force at the start of the age. For full years of age, there is no assumption regarding the distribution of deaths, ignoring decrements other than death. For partial ages, deaths may be assumed to be distributed according to the Balducci Hypothesis or the uniform distribution of deaths, depending on the exposure method used. Also known as the **actuarial** method.

Annual Force Method: Calculates the average force for an age as the number of deaths for the age divided by the amount of time all lives are in force during the year of age. The mortality rate is then calculated from the annual force, using an exponential formula. For full years of age, the distribution of deaths is assumed to be continuous and non-skewed. For partial ages, the force of mortality is assumed to be constant over the year. Also known as the **exact** method or **constant force** method. This is closely related to the **daily exposure** method, which calculates exposure in days: There is no practical difference between continuous and daily, as time of death has never been used.

Fractional Rate Method: Calculates the fractional mortality rate for a fractional age as the number of deaths for the fractional age divided by the number of lives in force at the start of the fractional age. The annual mortality rate is then calculated from the fractional mortality rates.

Fractional Force Method: Calculates the fractional average force for a fractional age as the number of deaths for the fractional age divided by the amount of time all lives are in force during the fractional year of age. The annual mortality rate is then calculated from the sum of fractional forces, using an exponential formula.

Annual Rate Method with traditional exposure: calculates the annual mortality rate for a partial age assigning exposure for deaths into the partial age that death occurs assuming that deaths are distributed according to the Balducci Hypothesis. Also referred to as the **traditional exposure method** or just the **traditional method**.

Annual Rate Method with distributed exposure: calculates the annual mortality rate for a partial age at the start of a rate year by assigning exposure for deaths only to the end of the partial age. The remaining exposure for those deaths, which runs from the end of the partial age to the end of the rate year, is distributed to the following partial age that completes the rate year. This method of assigning exposure is consistent with the uniform distribution of deaths. Also referred to as the **distributed exposure method** or just the **distributed method**.

Hybrid Annual Rate Method: This method uses traditional exposure for the partial age at the start of a study and distributed exposure for the partial age at the end of the study. Also referred to as the **hybrid method**.

Constant Force Method: calculates the average force for a partial age assuming the force is constant over the year. Also referred to as the **annual force method with unweighted exposure**.

Linear Force Method: calculates the average force for a partial age assuming the force is linear over the year. Also referred to as the **annual force method with weighted exposure**.

§8.1 Notation

Mortality Rate:

q_x: The annual mortality rate for a full year of age, from exact age x to $x+1$.

${}_fq_{x+s}$: The fractional mortality rate for a fractional age from exact age $x+s$ to $x+s+f$.

$q_{x+s,f}$: The annual mortality rate for a partial age from exact age $x+s$ to $x+s+f$.

$q_{x+s,f}^{BH}$: The annual mortality rate for a partial age from exact age $x+s$ to $x+s+f$, calculated using the traditional exposure method which assumes that deaths are distributed by the Balducci Hypothesis.

$q_{x+s,f}^{UD}$: The annual mortality rate for a partial age from exact age $x+s$ to $x+s+f$, calculated using the distributed exposure method which assumes that deaths are uniformly distributed.

$q_{x+s,f}^{CF}$: The annual mortality rate for a partial age from exact age $x+s$ to $x+s+f$, calculated using the annual force method assuming the force is constant over the year.

$q_{x+s,f,CY}$: The annual mortality rate for a partial age from exact age $x+s$ to $x+s+f$ for cohort year CY.

Average Force of Mortality:

$\bar{\mu}_x$: The annual average force of mortality for a full year of age from exact age x to $x+1$.

${}_f\bar{\mu}_{x+s}$: The fractional average force of mortality for a fractional age from exact age $x+s$ to $x+s+f$.

${}_f\bar{\mu}_{x+s}^{LF}$: The fractional average force of mortality for a fractional age from exact age $x+s$ to $x+s+f$, calculated assuming the linear force distribution.

$\bar{\mu}_{x+s,f}$: The annual average force of mortality for a partial age from exact age $x+s$ to $x+s+f$.

$\bar{\mu}_{x+s,f}^{CF}$: The annual average force of mortality for a partial age from exact age $x+s$ to $x+s+f$, calculated using the annual force method assuming the force is constant over the year.

$\bar{\mu}_{x+s,f}^{LF}$: The annual average force of mortality for a partial age from exact age $x+s$ to $x+s+f$, calculated using the annual force method assuming the force changes linearly over the year.

Related Force of Mortality:

μ_{x+t}: The annualized instantaneous force of mortality at exact age $x+t$

μ_{x+t}^{LF}: The annualized instantaneous force of mortality at exact age $x+t$, assuming the linear force distribution.

μ_{x+t}^{BLF}: The annualized instantaneous force of mortality at exact age $x+t$, assuming the bi-linear force distribution.

$\Delta\mu_x$: The increase in the force of mortality, or gradient, from exact age x to $x+1$.

Δ_x: The relative increase in the force of mortality, most often referred to as the **relative gradient**, from exact age x to $x+1$.

Deaths:

d_x: The number of deaths in a full year of age from exact age x to $x+1$.

${}_f d_{x+s}$: The number of deaths in a fractional age from exact age $x+s$ to $x+s+f$.

$d_{x+s,f}$: The number of deaths in a partial age from exact age $x+s$ to $x+s+f$.

$d_{x,f,CY}$: The number of deaths in a partial age from exact age $x+s$ to $x+s+f$ for cohort year CY.

Rate Exposure:

E_x: The annual exposure for a full year of age from exact age x to $x+1$ with survivors and deaths assigned 1 year of exposure.

${}_f E_{x+s}$: The fractional exposure for a fractional age from exact age $x+s$ to $x+s+f$, with survivors and deaths assigned 1 fractional period of exposure.

$E_{x+s,f}$: The annual exposure for a partial age from exact age $x+s$ to $x+s+f$.

$E_{x+s,f}^{BH}$: The annual exposure for a partial age from exact age $x+s$ to $x+s+f$, calculated using the traditional exposure method which assigns the remaining exposure for deaths into the partial age in which death occurs.

$E_{x+s,f}^{UD}$: The annual exposure for a partial age from exact age $x+s$ to $x+s+f$, calculated using the distributed exposure method which assigns remaining exposure for deaths into the partial ages following death.

$E_{x+s,f,CY}$: The annual exposure for a partial age from exact age $x+s$ to $x+s+f$ for cohort year CY.

Force Exposure:

E_x^F: The annual exposure for force for a full year of age from exact age x to $x+1$ with deaths assigned exposure in years from the start of the age up to the date of death.

${}_f E_{x+s}^F$: The fractional exposure for force for a fractional age from exact age $x+s$ to $x+s+f$ with deaths assigned exposure in fractional years from the start of the fractional age up to the date of death.

$E_{x+s,f}^F$: The annual exposure for force for a partial age for lives from exact age $x+s$ to $x+s+f$ with deaths assigned exposure in years from the start of the partial age up to the date of death.

$E_{x+s,f}^{CF}$: The annual exposure for force for a partial age for lives from exact age $x+s$ to $x+s+f$ with deaths assigned exposure in years from the start of the partial age up to the date of death.

$E_{x+s,f}^{LF}$: The annual exposure for force for a partial age for lives from exact age $x+s$ to $x+s+f$ with deaths assigned exposure in years from the start of the partial age up to the date of death, weighted for the distribution of forces over the year assuming the linear force distribution.

Exposure Weights:

$\alpha_{x+s,f}$: The partial exposure weight, i.e. $E_{x+s,f}$ divided by the total exposure for the full year of age for a partial age between exact age $x+s$ and $x+s+f$.

${}_f\alpha^F_{x+s}$: The fractional exposure weight, i.e. ${}_fE^F_{x+s}$ divided by the total exposure for the full year of age, for a partial age between exact age $x+s$ and $x+s+f$.

$\alpha^F_{x+s,f}$: The partial exposure weight, i.e. $E^F_{x+s,f}$ divided by the total exposure for the full year of age, for a partial age between exact age $x+s$ and $x+s+f$.

Annual Rate Error for a Partial Age:

$\varepsilon_{x+s,f}$: The error in the annual rate for a partial age from exact age $x+s$ to $x+s+f$.

$\varepsilon^G_{x+s,f}$: The error in the annual rate for a partial age from exact age $x+s$ to $x+s+f$, using the generic study method G.

$\varepsilon^{BH}_{x+s,f}$: The error in the annual rate for a partial age from exact age $x+s$ to $x+s+f$, using the traditional exposure method.

$\varepsilon^{UD}_{x+s,f}$: The error in the annual rate for a partial age from exact age $x+s$ to $x+s+f$, using the distributed exposure method.

$\varepsilon^{CF}_{x+s,f}$: The error in the annual rate for a partial age from exact age $x+s$ to $x+s+f$, using the annual force method which assumes the force is constant over the year.

Annual Force Error for a Partial Age:

$\xi_{x+s,f}$: The error in the annual force for a partial age from exact age $x+s$ to $x+s+f$, exclusively used with the annual force method.

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